ON THE COMPLEMENTED SUBSPACES OF THE SCHREIER SPACES

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Abstract. It is shown that for every $1 \leq \xi < \omega$ the Schreier space X^ξ admits a set of continuum cardinality whose elements are mutually incomparable complemented subspaces spanned by subsequences of (e_n^ξ), the natural Schauder basis of X^ξ. It is also shown that there exists a complemented subspace spanned by a block basis of (e_n^ξ), which is not isomorphic to a subspace generated by a subsequence of (e_n^ζ), for every $0 \leq \zeta \leq \xi$. Finally, an example is given of an uncomplemented subspace of X^ξ which is spanned by a block basis of (e_n^ξ).

1. Introduction

The Schreier families $\{S_\xi\}_{\xi<\omega_1}$ of finite subsets of positive integers (the precise definition is given in the next section), introduced in [1], have played a central role in the development of modern Banach space theory. We mention the use of Schreier families in the construction of mixed Tsirelson spaces which are asymptotic ℓ_1 and arbitrarily distortable [3]. The distortion of mixed Tsirelson spaces has been extensively studied in [2]. In that paper as well as in [14], the moduli $(\delta_\alpha)_{\alpha<\omega_1}$ were introduced measuring the complexity of the asymptotic ℓ_1 structure of a Banach space. The definitions of those moduli also involve the Schreier families. Other applications can be found in [4] and [6] where the Schreier families form the main tool for determining the structure of those convex combinations of a weakly null sequence that tend to zero in norm, or are equivalent to the unit vector basis of c_0. For applications of the Schreier families in the construction of hereditarily indecomposable Banach spaces, we refer to [3] and [5].

A notion companion to the Schreier families is that of the Schreier spaces. These are Banach spaces whose norm is related to a corresponding Schreier family. More precisely, for every countable ordinal ξ, we define a norm $\|\cdot\|_\xi$ on c_{00}, the space of finitely supported real valued sequences, in the following manner: Given $x = \big(x(n)\big) \in c_{00}$ define

$$\|x\|_\xi = \sup_{F\in S_\xi} \sum_{n\in F} |x(n)|.$$

Date: July 3, 1999.
1991 *Mathematics Subject Classification.* Primary: 46B03. Secondary: 06A07, 03E10.
Key words and phrases. Complemented subspace, Schreier sets.

X^ξ, the Screier space of order ξ, is the completion of c_{00} under the norm $\|\cdot\|_\xi$. $X^0 = c_0$, the Banach space of null sequences. X^1 was first considered by Schreier [15] in order to provide an example of a weakly null sequence without Cesaro summable subsequence. It is proven in [1] that the natural Schauder basis (e_n^ξ) of X^ξ is 1-unconditional and shrinking. X^1 has been studied in [13] where it is shown that every quotient of X^1 is c_0-saturated. That is, every infinite dimensional subspace contains a further subspace isomorphic to c_0.

Given M, an infinite subset of $\mathbb{N}$, we let X_M^ξ denote the closed linear subspace of X^ξ spanned by the subsequence $(e_n^\xi)_{n\in M}$. For an element $x \in X^\xi$, $x = \sum_{n\in\mathbb{N}} a_n e_n^\xi$, we set $\|x\|_0 = \sup_{n\in\mathbb{N}} |a_n|$. The main result of this paper is the following

Theorem 1.1. *Let $L = (l_n)$, $M = (m_n)$ be infinite subsets of $\mathbb{N}$, and let $\xi < \omega$. The following are equivalent:*

1. *There exist a bounded linear operator $T\colon X_L^\xi \to X_M^\xi$ and $\delta > 0$ such that $\|T(e_l^\xi)\|_0 > \delta$, for all $l \in L$.*
2. *$(e_{l_n}^\zeta)$ dominates $(e_{m_n}^\zeta)$, for every $\zeta \le \xi$.*
3. *$(e_{l_n}^\xi)$ dominates $(e_{m_n}^\xi)$.*

We recall here that a basic sequence (x_n) in some Banach space X is said to *dominate* the basic sequence (y_n) in the Banach space Y, if there exists a constant $C > 0$ so that $\|\sum_{i=1}^n a_i y_i\| \le C\|\sum_{i=1}^n a_i x_i\|$, for every $n \in \mathbb{N}$ and all scalar sequences $(a_i)_{i=1}^n$. Equivalently, (x_n) dominates (y_n) if there exists a bounded linear operator T from the closed linear span of (x_n) into the closed linear span of (y_n) so that $T(x_n) = y_n$, for every $n \in \mathbb{N}$. The sequences (x_n) and (y_n) are *equivalent* if each one of them dominates the other.

As an immediate consequence of Theorem 1.1 we obtain

Corollary 1.2. *Let $\xi < \omega$ and $L = (l_n)$, $M = (m_n)$ be infinite subsets of $\mathbb{N}$.*

1. *If X_L^ξ is isomorphic to a subspace of X_M^ξ then $(e_{l_n}^\xi)$ dominates $(e_{m_n}^\xi)$. Consequently, X_L^ξ is isomorphic to X_M^ξ if, and only if, $(e_{l_n}^\xi)$ is equivalent to $(e_{m_n}^\xi)$.*
2. *If X_L^ξ is isomorphic to X_M^ξ, then X_L^ζ is isomorphic to X_M^ζ, for every $\zeta \le \xi$.*
3. *Suppose that $(e_{l_n}^\xi)$ dominates a permutation of $(e_{m_n}^\xi)$. Then $(e_{l_n}^\xi)$ dominates $(e_{m_n}^\xi)$.*

Theorem 1.1 combined with elementary descriptive set theory yields our next result on the structure of the subsequences of (e_n^ξ), $\xi < \omega$. We recall here that the Banach spaces X and Y are incomparable if neither of them is isomorphic to a closed linear subspace of the other.

Theorem 1.3. *For every $\xi < \omega$ there exists a set $\mathcal{A}$ (depending on ξ) consisting of infinite subsets of $\mathbb{N}$ and satisfying the following properties*

1. *The cardinality of $\mathcal{A}$ is equal to the continuum.*
2. *For every pair (L, M) of distinct elements of $\mathcal{A}$, the spaces X_L^ξ and X_M^ξ are incomparable.*

The proofs of the aforementioned results are given in the third section of our paper. In the fourth section we deal with complemented subspaces of X^ξ spanned by block bases of (e_n^ξ). We show that there exists a block basis of (e_n^ξ) spanning a complemented subspace of X^ξ which is not isomorphic to X_M^ζ, for all $0 \leq \zeta \leq \xi$ and every infinite subset M of $\mathbb{N}$. We also show that there exists a block basis of (e_n^ξ) spanning a subspace which is not complemented in X^ξ.

The problem of the isomorphic classification of the complemented subspaces of X^ξ, even for block subspaces, seems rather difficult.

Part of the research for this paper was conducted while the second author visited the University of Texas at Austin. The second author thanks the Department of Mathematics there, especially the Banach space group, for making the visit possible. Thanks are also due to Ted Odell for several conversations regarding the results contained herein.

2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [11]. In this section we shall review some of the necessary concepts. We shall also review two important hierarchies, the Schreier hierarchy [1] and the repeated averages hierarchy [4]. Finally we shall state some fundamental results from descriptive set theory which will be widely used in the sequel. For a detailed study of descriptive set theory we refer to [9].

We first indicate some special notation that we will be using. A sequence $(x_n)_{n=1}^\infty$ of elements of an arbitrary set will be conveniently denoted by (x_n). Given M, a subset of $\mathbb{N}$, $[M]^{<\infty}$ denotes the set of all finite subsets of M, while $[M]$ stands for the set of all infinite subsets of M. If $M \in [\mathbb{N}]$, then the notation $M = (m_n)$ indicates that $M = \{m_1 < m_2 < \cdots\}$. Let E, F be finite sets of integers. We shall adopt the notation $E < F$ to denote the relation $\max E < \min F$. If $x = \big(x(n)\big)$ belongs to c_{00}, the space of finitely supported real valued sequences, and $F \in [\mathbb{N}]^{<\infty}$, then $x(F) = \sum_{n \in F} x(n)$, and $|x|(F) = \sum_{n \in F} |x(n)|$.

All Banach spaces considered throughout this paper are real. ℓ_1 denotes the Banach space of the absolutely summable sequences under the norm given by the sum of the absolute values of the coordinates. c_0 is the Banach space of the null sequences under the norm given by the maximum of the absolute values of the coordinates. By the term *"subspace"* of a Banach space we shall always mean a closed linear subspace. A subspace Y of the

Banach space X is said to be *complemented* if it is the range of a bounded linear projection on X.

We next recall that if (x_n) is a sequence in some normed linear space, then the sequence (y_n) is called a *block subsequence* (resp. *convex block subsequence*) of (x_n), if there exist sets $F_i \subset \mathbb{N}$ with $F_1 < F_2 < \cdots$ and a sequence (a_i) of scalars (resp. non-negative scalars such that $\sum_{n \in F_i} a_n = 1$, for every $i \in \mathbb{N}$) such that for every $i \in \mathbb{N}$, $y_i = \sum_{n \in F_i} a_n x_n$. We then denote by $suppy_i$, the support of y_i, that is the set $\{n \in F_i : |a_n| > 0\}$. We shall also adopt the notation $y_1 < y_2 < \cdots$ to indicate that (y_n) is a block subsequence of (x_n). In case (x_n) is Schauder basic, then (y_n) will be called a *block basis* (resp. *convex block basis*) of (x_n).

Next we review the definition and some basic properties of the Schreier families $\{S_\xi\}_{\xi<\omega_1}$ [1]. The Schreier families are defined by transfinite induction as follows:

$$S_0 = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}.$$

Suppose S_ζ has been defined for every $\zeta < \xi$. If ξ is a successor ordinal, say $\xi = \zeta + 1$, we set

$$S_\xi = \{\cup_{i=1}^n F_i : n \in \mathbb{N}, n \leq \min F_1, F_1 < \cdots < F_n, F_i \in S_\zeta (i \leq n)\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal, let (ξ_n) be a preassigned increasing sequence of successor ordinals whose limit is ξ. We set

$$S_\xi = \cup_{n=1}^\infty \{F \in S_{\xi_n} : n \leq \min F\} \cup \{\emptyset\}.$$

Given $M \in [\mathbb{N}]$ we denote by $S_\xi[M]$ the family $\{F : F \in S_\xi, F \subset M\}$.

An important property shared by the Schreier families is that they are spreading: If $\{p_1, \cdots, p_k\} \in S_\xi$, $p_1 < \cdots < p_k$, and $q_1 < \cdots < q_k$ are so that $p_i \leq q_i$ for all $i \leq k$, then $\{q_1, \cdots, q_k\} \in S_\xi$.

Of particular interest are the maximal (under inclusion) members of S_ξ. The following lemma concerning those sets is proved in [8].

Lemma 2.1. *Let $M \in [\mathbb{N}]$ and $\xi < \omega_1$. Then there exists a (necessarily) unique sequence $\{F_n^\xi(M)\}_{n=1}^\infty$ of successive maximal S_ξ sets so that $M = \cup_{n=1}^\infty F_n^\xi(M)$.*

Remark . *The following stability properties of $\{F_n^\xi(M)\}_{n=1}^\infty$ are easily verified:*

1. *If $k_1 < k_2 < \cdots$ and $N = \cup_{n=1}^\infty F_{k_n}^\xi(M)$, then $F_n^\xi(N) = F_{k_n}^\xi(M)$, for all $n \in \mathbb{N}$.*
2. *Let $M = (m_i)$ and $N = (n_i)$ be infinite subsets of $\mathbb{N}$. Assume that for some $p \in \mathbb{N}$, $m_i = n_i$ for all $i \leq p$. If $F_k^\xi(M)$ is contained in $\{m_i : i \leq p\}$, then $F_i^\xi(M) = F_i^\xi(N)$ for all $i \leq k$.*

In the sequel we shall make use of the following

Lemma 2.2. *Let $M \in [\mathbb{N}]$, $L \in [M]$ and $\xi < \omega$. Then $\max F_1^\xi(M) \leq \max F_1^\xi(L)$.*

Proof. Suppose $L = (l_i)$ and $M = (m_i)$. We prove the assertion of the lemma by induction on ξ. The case $\xi = 0$ is trivial. Assume now that $\xi \geq 1$ and that the assertion holds for $\xi - 1$ and all P, Q with $Q \in [P]$.

For an arbitrary $P \in [\mathbb{N}]$, we set $P_1 = P$ and

$$P_i = \{p \in P : \, p > \max F_{i-1}^{\xi-1}(P)\}, \, i \geq 2.$$

We observe that $F_i^{\xi-1}(P) = F_1^{\xi-1}(P_i)$, for all $i \in \mathbb{N}$. We also have that $F_1^{\xi}(P) = \cup_{i=1}^{p_1} F_1^{\xi-1}(P_i)$, where $p_1 = \min P_1$. It follows now, by the induction hypothesis, that $L_i \in [M_i]$, for all $i \in \mathbb{N}$. Therefore,

$$\max F_1^{\xi}(M) = \max F_1^{\xi-1}(M_{m_1}) \leq \max F_1^{\xi-1}(L_{m_1}) \leq \max F_1^{\xi}(L)$$

as $m_1 \leq l_1$. The proof of the lemma is now complete. $\square$

We now pass to the definition of the repeated averages hierarchy introduced in [4]. We let (e_n) denote the unit vector basis of c_{00}. For every countable ordinal ξ and every $M \in [\mathbb{N}]$, we define a convex block subsequence $(\xi_n^M)_{n=1}^{\infty}$ of (e_n) by transfinite induction on ξ in the following manner: If $\xi = 0$, then $\xi_n^M = e_{m_n}$, for all $n \in \mathbb{N}$, where $M = (m_n)$.

Assume that $(\zeta_n^M)_{n=1}^{\infty}$ has been defined for all $\zeta < \xi$ and $M \in [\mathbb{N}]$. Let $\xi = \zeta + 1$. Set

$$\xi_1^M = \frac{1}{m_1} \sum_{i=1}^{m_1} \zeta_i^M$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \cdots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max supp \xi_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = \frac{1}{k_n} \sum_{i=1}^{k_n} \zeta_i^{M_n}.$$

If ξ is a limit ordinal, let $(\xi_n + 1)$ be the sequence of ordinals associated to ξ in the definition of S_ξ, and let also $M \in [\mathbb{N}]$. Define

$$\xi_1^M = [\xi_{m_1} + 1]_1^M$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \cdots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max supp \xi_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = [\xi_{k_n} + 1]_1^{M_n}.$$

The inductive definition of $(\xi_n^M)_{n=1}^{\infty}$, $M \in [\mathbb{N}]$ is now complete. The following properties are established in [4].

P1: $(\xi_n^M)_{n=1}^{\infty}$ is a convex block subsequence of (e_n) and $M = \cup_{n=1}^{\infty} supp \xi_n^M$ for all $M \in [\mathbb{N}]$ and $\xi < \omega_1$.

P2: $supp \xi_n^M \in S_\xi$, for all $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $n \in \mathbb{N}$.

P3: If $M, N \in [\mathbb{N}]$, $\xi < \omega_1$, and $supp\xi_i^M = supp\xi_i^N$, for $i \leq k$, then $\xi_i^M = \xi_i^N$ for $i \leq k$.

P4: If $\xi < \omega_1$, $\{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$, and $\{L_k : k \in \mathbb{N}\} \subset [\mathbb{N}]$, are such that $supp\xi_{n_i}^{L_i} < supp\xi_{n_{i+1}}^{L_{i+1}}$, for all $i \in \mathbb{N}$, then letting $L = \cup_{i=1}^{\infty} supp\xi_{n_i}^{L_i}$, we have that $\xi_i^L = \xi_{n_i}^{L_i}$, for all $i \in \mathbb{N}$.

Properties **P3** and **P4** are called stability properties of the hierarchy $\{(\xi_n^M)_{n=1}^{\infty} : M \in [\mathbb{N}]\}$. It is easily seen, by induction, that $supp\xi_n^M = F_n^{\xi}(M)$, for every $\xi < \omega$, all $M \in [\mathbb{N}]$ and $n \in \mathbb{N}$.

In the next lemma we show that for $\xi < \omega$ and $M \in [\mathbb{N}]$ the sequence (ξ_n^M), considered as a sequence in X^{ξ}, is equivalent to the unit vector basis of c_0. Moreover, the equivalence constant depends only on ξ.

Lemma 2.3. $\|\sum_{i=1}^n \xi_i^M\|_{\xi} \leq \xi + 1$, *for every* $M \in [\mathbb{N}]$, $n \in \mathbb{N}$, *and* $\xi < \omega$.

Proof. By induction on ξ. The case $\xi = 0$ is trivial. Assume the assertion holds for $\xi - 1$. Let $G \in S_{\xi}$. We shall show that $\sum_{i=1}^n \xi_i^M(G) \leq \xi + 1$, for every $M \in [\mathbb{N}]$ and $n \in \mathbb{N}$. To this end choose $G_1 < \cdots < G_l$, successive members of $S_{\xi-1}$ so that $l \leq \min G$ and $G = \cup_{i=1}^l G_i$. Let also $\{i_1, \dots i_p\}$ be an enumeration of the set $\{i \leq n : F_i^{\xi}(M) \cap G \neq \emptyset\}$. We define

$$L = \cup_{t=1}^p F_{i_t}^{\xi}(M) \cup \{m \in M : m > \max F_{i_p}^{\xi}(M)\}$$

and observe that $F_{i_t}^{\xi}(M) = \cup_{j=r_{t-1}+1}^{r_t} F_j^{\xi-1}(L)$, for all $t \leq p$, where, $r_0 = 0 < r_1 < \cdots < r_p$ are chosen so that $r_t - r_{t-1} = \min F_{i_t}^{\xi}(M)$, for all $t \leq p$. Therefore,

$$\xi_{i_t}^M = \frac{1}{\min F_{i_t}^{\xi}(M)} \sum_{j=r_{t-1}+1}^{r_t} (\xi - 1)_j^L,\ t \leq p,$$

and thus

$$\begin{aligned}
\sum_{t=2}^p \xi_{i_t}^M(G) &= \sum_{t=2}^p \sum_{s=1}^l \frac{1}{\min F_{i_t}^{\xi}(M)} \sum_{j=r_{t-1}+1}^{r_t} (\xi-1)_j^L(G_s) \\
&= \sum_{s=1}^l \sum_{t=2}^p \frac{1}{\min F_{i_t}^{\xi}(M)} \sum_{j=r_{t-1}+1}^{r_t} (\xi-1)_j^L(G_s) \\
&\leq \sum_{s=1}^l \frac{1}{\min F_{i_2}^{\xi}(M)} \left(\sum_{t=2}^p \sum_{j=r_{t-1}+1}^{r_t} (\xi-1)_j^L(G_s) \right) \\
&\leq \sum_{s=1}^l \frac{1}{\min F_{i_2}^{\xi}(M)} \xi \text{ (by the induction hypothesis)} \\
&= \frac{l}{\min F_{i_2}^{\xi}(M)} \xi \\
&\leq \xi \text{ as } \ l \leq \min G \leq \max F_{i_1}^{\xi}(M) < \min F_{i_2}^{\xi}(M).
\end{aligned}$$

Finally, $\sum_{t=1}^{p} \xi_{i_t}^M(G) \leq 1+\xi$ and hence $\sum_{i=1}^{n} \xi_i^M(G) \leq 1+\xi$. We conclude, since $G \in S_\xi$ was arbitrary, that $\| \sum_{i=1}^{n} \xi_i^M \|_\xi \leq \xi + 1$, as claimed. □

Let now $M \in [\mathbb{N}]$. By identifying elements of $[M]$ with their indicator functions, $[M]$ can be endowed with the topology of pointwise convergence. It is not difficult to see that $[M]$ is then homeomorphic to a G_δ subset of the Cantor set $\{0,1\}^{\mathbb{N}}$, and thus it is a zero-dimensional Polish space. Further, $[M]$ is perfect (that is contains no isolated points) and every compact subset of $[M]$ is nowhere dense. It is then a classical result that $[M]$, endowed with the topology of pointwise convergence, is homeomorphic to the space of irrational numbers with the ordinary topology. It is worthwhile to note here that the family

$$\{W(p_1,\dots,p_k) : k \in \mathbb{N},\, p_1 < \cdots < p_k,\, p_i \in M,\, i \leq k\}$$

where $W(p_1,\dots,p_k) = \{L \in [M],\, L = (l_i) : l_i = p_i,\, i \leq k\}$, forms a basis of clopen subsets for the topology of the pointwise convergence in $[M]$.

3. Proofs of the main results

This section is devoted to the proofs of Theorems 1.1 and 1.3.

Definition 3.1. *Let $\xi < \omega_1$ and $A \in [\mathbb{N}]^{<\infty}$. We set*

$$\tau_\xi(A) = \max\Big\{ n \in \mathbb{N} : \, A \cap F_n^\xi\big(A \cup \{m \in \mathbb{N} : \, m > \max A\}\big) \neq \emptyset \Big\}.$$

We observe that $\tau_\xi(A)$ remains invariant if $\{m \in \mathbb{N} : m > \max A\}$ is replaced by $\{m \in M : m > \max A\}$, $M \in [\mathbb{N}]$, in Definition 3.1. The quantity $\tau_\xi(A)$ is important for our purposes since it will enable us state a criterion for determining whether or not the sequence $(e_{l_n}^\xi)$ dominates $(e_{m_n}^\xi)$, where $L = (l_n)$ and $M = (m_n)$ belong to $[\mathbb{N}]$. Our next lemma describes some permanence properties of $\tau_\xi(A)$.

Lemma 3.2. *Let $\xi < \omega_1$ and A, B belong to $[\mathbb{N}]^{<\infty}$.*

1. *If $A \subset B$ then $\tau_\xi(A) \leq \tau_\xi(B)$.*
2. *If $A < B$ then $\tau_\xi(A \cup B) \leq \tau_\xi(A) + \tau_\xi(B)$.*
3. *If $A = \{a_1 < \cdots < a_n\}$, $B = \{b_1 < \cdots < b_n\}$, $n \in \mathbb{N}$, and $a_i \leq b_i$ for $i \leq n$, then $\tau_\xi(B) \leq \tau_\xi(A)$.*
4. *Assume that $A = \cup_{i=1}^n A_i$, $B = \cup_{i=1}^n B_i$, where $n \in \mathbb{N}$, and $A_1 < \cdots < A_n$, $B_1 < \cdots < B_n$ are maximal members of S_ξ. If $\min A_i \leq \min B_i$, for all $i \leq n$, then $\tau_{\xi+1}(B) \leq \tau_{\xi+1}(A)$.*
5. *Assume that $A = \cup_{i=1}^n A_i$ for some $n \in \mathbb{N}$. Then*

$$\tau_\xi(A) \leq \Big(\sum_{i=1}^{n} \tau_\xi(A_i)\Big)(\xi+1) + 1$$

for any $\xi < \omega$.

Proof. The first two properties are immediate consequences of Definition 3.1. The third property follows because S_ξ is spreading. Let us show that 4. holds. This is accomplished by induction on n. The case $n = 1$ is easy because $\tau_{\xi+1}(B) = \tau_{\xi+1}(A) = 1$. Assuming the assertion true for all $k < n$, we set $k_1 = \min A_1$ and $l_1 = \min B_1$. In case $l_1 \geq n$, we obtain that $B \in S_{\xi+1}$. Thus $\tau_{\xi+1}(B) = 1$ and hence the assertion holds.

Next suppose that $l_1 < n$. It follows that $\cup_{i=1}^{k_1} A_i$ and $\cup_{i=1}^{l_1} B_i$ are maximal $S_{\xi+1}$ sets. On the other hand, because $n - l_1 < n$, the induction hypothesis yields that $\tau_{\xi+1}(\cup_{i=l_1+1}^{n} B_i) \leq \tau_{\xi+1}(\cup_{i=l_1+1}^{n} A_i)$. But also, $k_1 \leq l_1$ and so property 1. yields that $\tau_{\xi+1}(\cup_{i=l_1+1}^{n} A_i) \leq \tau_{\xi+1}(\cup_{i=k_1+1}^{n} A_i)$. The proof is complete since $\tau_{\xi+1}(B) = 1 + \tau_{\xi+1}(\cup_{i=l_1+1}^{n} B_i)$, while $\tau_{\xi+1}(A) = 1 + \tau_{\xi+1}(\cup_{i=k_1+1}^{n} A_i)$.

We now prove 5. Let $k = \tau_\xi(A)$ and $M = A \cup \{m \in \mathbb{N} : m > \max A\}$. Denote $\sum_{j=1}^{k-1} \xi_j^M$ by x. By Lemma 2.3, $\|x\|_\xi \leq \xi + 1$. Hence

$$\begin{aligned} k - 1 = x(A) &\leq \sum_{i=1}^{n} x(A_i) \\ &\leq (\sum_{i=1}^{n} \tau_\xi(A_i))\|x\|_\xi \\ &\leq (\sum_{i=1}^{n} \tau_\xi(A_i))(\xi + 1), \end{aligned}$$

from which the result follows. $\square$

Definition 3.3. *Let $\xi < \omega_1$ and $L = (l_n)$, $M = (m_n)$ belong to $[\mathbb{N}]$. Define*

$$d_\xi(L, M) = \sup\{\tau_\xi(\phi^{-1}A) : A \in S_\xi[M]\}.$$

Where $\phi\colon L \to M$ is the natural bijection $\phi(l_n) = m_n$, for all $n \in \mathbb{N}$.

The reason we introduced the quantity $d_\xi(L, M)$ is justified by our next lemma.

Lemma 3.4. *Let $\xi < \omega$ and $L = (l_n)$, $M = (m_n)$ belong to $[\mathbb{N}]$. Then $(e_{l_n}^\xi)$ dominates $(e_{m_n}^\xi)$ if and only if, $d_\xi(L, M)$ is finite.*

Proof. Suppose first that $d_\xi(L, M) = p < \infty$. Let $(a_i)_{i=1}^n$ be scalars and choose $F \in S_\xi[M]$ so that $\sum_{i\in H} |a_i| = \|\sum_{i=1}^n a_i e_{m_i}^\xi\|$, where we have set $H = \{i \leq n : m_i \in F\}$. It follows, by our assumption, that we can find $G_1 < \cdots < G_p$ successive $S_\xi[L]$ sets so that $\phi^{-1}F \subset \cup_{j=1}^p G_j$. We now set $H_j = \{i \in H : l_i \in G_j\}$, for all $j \leq p$. It is clear that $H = \cup_{j=1}^p H_j$ and moreover, $\{l_i : i \in H_j\}$ belongs to $S_\xi[L]$. Finally,

$$\sum_{i\in H} |a_i| = \sum_{j=1}^{p} \sum_{i\in H_j} |a_i| \leq p \left\| \sum_{i=1}^{n} a_i e_{l_i}^\xi \right\|.$$

Thus, $\| \sum_{i=1}^n a_i e^\xi_{m_i} \| \leq p \| \sum_{i=1}^n a_i e^\xi_{l_i} \|$.

Conversely, assume that $(e^\xi_{l_n})$ C-dominates $(e^\xi_{m_n})$ and let $F \in S_\xi[M]$. Suppose that $\tau_\xi(\phi^{-1}F) = k$. It follows that there exist $G_1 < \cdots < G_{k-1}$, successive maximal $S_\xi[L]$ sets so that $\cup_{i=1}^{k-1} G_i \subset \phi^{-1}F$. Put

$$Q = \cup_{i=1}^{k-1} G_i \cup \{l \in L : l > \max G_{k-1}\}.$$

We may write $\xi_i^Q = \sum_{j \in G_i} a^i_j e^\xi_j$, with $\sum_{j \in G_i} a^i_j = 1$, for all $i \leq k-1$. If we apply Lemma 2.3, we obtain

$$\begin{aligned} C(\xi+1) &\geq C \left\| \sum_{i=1}^{k-1} \xi_i^Q \right\|_\xi = C \left\| \sum_{i=1}^{k-1} \sum_{j \in G_i} a^i_j e^\xi_j \right\| \\ &\geq \left\| \sum_{i=1}^{k-1} \sum_{j \in G_i} a^i_j e^\xi_{\phi(j)} \right\| \geq k-1, \end{aligned}$$

as $\cup_{i=1}^{k-1} \{\phi(j) : j \in G_i\} \subset F$ and $\sum_{j \in G_i} a^i_j = 1$. Hence, $k \leq C(\xi+1)+1$ which implies that $d_\xi(L,M) \leq C(\xi+1)+1$ as F was an arbitrary $S_\xi[M]$ set. □

We shall next show that (e^ξ_n) has "many" non-equivalent subsequences.

Lemma 3.5. *Let $1 \leq \xi < \omega$, $N \in [\mathbb{N}]$ and set*

$$\mathcal{D} = \{(L,M) \in [N] \times [N] : d_\xi(L,M) = d_\xi(M,L) = \infty\}.$$

Then $\mathcal{D}$ is a G_δ dense subset of $[N] \times [N]$.

Proof. By Baire's theorem, it suffices to show that the sets $\{(L,M) \in [N] \times [N] : d_\xi(L,M) < \infty\}$ and $\{(L,M) \in [N] \times [N] : d_\xi(M,L) < \infty\}$ are first category F_σ subsets of $[N] \times [N]$. Indeed, we may write

$$\{(L,M) \in [N] \times [N] : d_\xi(L,M) < \infty\} = \cup_{n=1}^\infty \{(L,M) : d_\xi(L,M) \leq n\}.$$

It is easy to see that each set in the union is closed in $[N] \times [N]$ and thus it remains to show that $\{(L,M) \in [N] \times [N] : d_\xi(L,M) \leq n\}$ has empty interior in $[N] \times [N]$. If that were not the case, choose $\mathcal{U}$ and $\mathcal{V}$, non-empty basic clopen subsets of $[N]$ so that $\mathcal{U} \times \mathcal{V}$ is contained in $\{(L,M) \in [N] \times [N] : d_\xi(L,M) \leq n\}$. There exist $p_1 < \cdots < p_k$ in N so that $\mathcal{V} = W(p_1, \ldots, p_k)$. Fix $L \in \mathcal{U}$, $L = (l_i)$. If $M \in [N]$, $\min M > p_k$, let $P = \{p_1, \cdots, p_k\} \cup M$. Since $d_\xi(L,P) \leq n$, it follows that if $L_k = (l_{k+i})_{i=1}^\infty$, then $d_\xi(L_k, M) \leq n$. By Lemma 3.4 this implies that $(e^\xi_l)_{l \in L_k}$ is equivalent to the unit vector basis of ℓ_1 which is absurd. Arguing similarly, we also obtain that $\{(L,M) \in [N] \times [N] : d_\xi(M,L) < \infty\}$ is first category, F_σ subset of $[N] \times [N]$. □

We also need the following result which is a special case of a theorem by Mycielski [12] and Kuratowski [10] (cf. also [9]).

Proposition 3.6. *Let K be a perfect Polish space and G a G_δ dense subset of $K \times K$. There exists C, a subset of K homeomorphic to the Cantor set such that $C \times C \setminus \Delta \subset G$ (here Δ is the diagonal subset of $K \times K$).*

This result may be found in [9] (p. 129, Theorem 19.1) but we shall include a proof to be thorough.

Lemma 3.7. *Let K be Polish and G be an open dense subset of $K \times K$. Let also $(A_i)_{i=1}^n$ ($n \geq 2$) be a finite sequence of open non-empty subsets of K. Then for every $\epsilon > 0$ there exist $(B_i)_{i=1}^n$, open non-empty subsets of K, so that $diam B_i < \epsilon$, for all $i \leq n$ and $\overline{B_i} \times \overline{B_j} \subset G \cap (A_i \times A_j)$, for all $i \neq j$ in $\{1, \ldots, n\}$.*

Proof. By induction on n. Suppose first that $n = 2$. Since $(A_1 \times A_2) \cap G \neq \emptyset$, there exist C_1, C_2, open non-empty subsets of K whose diameters are smaller than ϵ, so that $\overline{C_1} \times \overline{C_2} \subset G \cap (A_1 \times A_2)$. Further, $(C_2 \times C_1) \cap G \neq \emptyset$ and thus there exist B_1, B_2, open non-empty subsets of K, so that $\overline{B_2} \times \overline{B_1} \subset G \cap (C_2 \times C_1)$. Of course B_1 and B_2 satisfy the conclusion of the lemma for $n = 2$.

Next assume $n > 2$ and that the result holds for $n - 1$. We can therefore choose $(C_i)_{i=1}^{n-1}$, open non-empty subsets of K with diameters smaller than ϵ, so that $\overline{C_i} \times \overline{C_j} \subset G \cap (A_i \times A_j)$, for all $i \neq j$ in $\{1, \ldots, n-1\}$. Next, set $A_{n,0} = A_n$ and choose, as in the case $n = 2$, $(B_i)_{i=1}^{n-1}$, $(A_{n,i})_{i=1}^{n-1}$, non-empty open subsets of K with diameters smaller than ϵ, so that $\overline{B_i} \times \overline{A_{n,i}} \subset G \cap (C_i \times A_{n,i-1})$ and $\overline{A_{n,i}} \times \overline{B_i} \subset G \cap (A_{n,i-1} \times C_i)$, for all $i \leq n-1$. Set $B_n = A_{n,n-1}$ and it is easy to check that $(B_i)_{i=1}^n$ is the desired sequence. $\square$

Proof of Proposition 3.6. Since K contains no isolated points, Δ is nowhere dense in $K \times K$. Hence, $G \cap (K \times K \setminus \Delta)$ is a G_δ dense subset of $K \times K$. We shall therefore assume, without loss of generality, that $G \cap \Delta = \emptyset$. Now let (G_n) be a decreasing sequence of open dense subsets of $K \times K$, whose intersection is G. We can assume that $G_n \cap \Delta = \emptyset$, for all $n \in \mathbb{N}$.

We shall construct a collection $\{U_\alpha : \alpha \in \{0,1\}^n,\, n \in \mathbb{N}\}$ of open non-empty subsets of K so that the following properties are satisfied for every $n \in \mathbb{N}$:

(i) $\overline{U_\alpha} \cap \overline{U_\beta} = \emptyset$, whenever $\alpha \neq \beta$ in $\{0,1\}^n$.

(ii) $\overline{U_\alpha} \subset U_\beta$, for all $\alpha \in \{0,1\}^n$ and every $\beta \in \{0,1\}^m$, ($m < n$), initial segment of α.

(iii) $diam U_\alpha < \frac{1}{n}$, for every $\alpha \in \{0,1\}^n$.

(iv) $\overline{U_\alpha} \times \overline{U_\beta} \subset G_n$, whenever $\alpha \neq \beta$ in $\{0,1\}^n$.

Once this is accomplished, we let

$$C = \{x \in K : \exists\, \alpha \in \{0,1\}^{\mathbb{N}},\, \{x\} = \cap_{n=1}^\infty U_{\alpha|n}\}.$$

Where $\alpha|n = (a_1, \ldots, a_n)$, if $\alpha = (a_i) \in \{0,1\}^{\mathbb{N}}$. It is a standard result that C is homeomorphic to the Cantor set. Property (iv) yields that C satisfies the conclusion of Proposition 3.6.

The construction is done by induction on $n \in \mathbb{N}$. For $n = 1$ choose W_0 and W_1, open non-empty subsets of K so that $W_0 \times W_1 \subset G_1$. W_0 and W_1 are disjoint since $G_1 \cap \Delta = \emptyset$. If we apply Lemma 3.7, for $\epsilon = 1$, on the dense open subset G_1 and the open sets W_0 and W_1, we shall find U_0, U_1, non-empty open subsets of K, satisfying properties (i)-(iv), for $n = 1$.

Now suppose that for every $k \leq n$ we have constructed $\{U_\alpha : \alpha \in \{0,1\}^k\}$, a collection of open non-empty subsets of K whose members satisfy properties (i)-(iv), for k. Let $\{d_1, \dots, d_p\}$, $p = 2^n$, be an enumeration of $\{0,1\}^n$. Another application of Lemma 3.7 yields W_{j0} and W_{j1}, non-empty open subsets of K, $j \leq p$, so that $\overline{W_{jr}} \times \overline{W_{js}} \subset (U_{d_j} \times U_{d_j}) \cap G_{n+1}$, for every $j \leq p$ and all pairs (r, s) of distinct elements of $\{0, 1\}$. It follows, since $G_{n+1} \cap \Delta = \emptyset$, that $W_{j0} \cap W_{j1} = \emptyset$. According to the induction hypothesis $\overline{U_{d_j}} \cap \overline{U_{d_i}} = \emptyset$, for all $i \neq j$ in $\{1, \dots, p\}$, and thus $\overline{W_{jr}} \cap \overline{W_{is}} = \emptyset$, for all $(j, r) \neq (i, s)$ in $\{1, \dots, p\} \times \{0, 1\}$.

We next apply Lemma 3.7, for $\epsilon = \frac{1}{n+1}$, on the family $\{W_{jr} : (j, r) \in \{1, \dots, p\} \times \{0, 1\}\}$ and the dense open subset G_{n+1}. We shall obtain $\{U_\alpha : \alpha \in \{0,1\}^{n+1}\}$, a collection of non-empty open subsets of K, so that $\overline{U_\alpha} \times \overline{U_\beta} \subset (W_{jr} \times W_{is}) \cap G_{n+1}$ whenever $\alpha = (d_j, r)$, $\beta = (d_i, s)$ and $(j, r) \neq (i, s)$ in $\{1, \dots, p\} \times \{0, 1\}$. Evidently, $\{U_\alpha : \alpha \in \{0,1\}^{n+1}\}$ satisfies properties (i)-(iv). The inductive step as well as the proof of the proposition are now complete. $\square$

Assuming we have proved Theorem 1.1 and Corollary 1.2, let us now show how to derive Theorem 1.3 from our previously obtained results.

Proof of Theorem 1.3. Let $\mathcal{D}$ be as in the statement of Lemma 3.5, where we have taken $N = \mathbb{N}$. We can apply Proposition 3.6 for the space $[\mathbb{N}]$ and the subset $\mathcal{D}$ to obtain $\mathcal{A} \subset [\mathbb{N}]$, homeomorphic to the Cantor set and such that $\mathcal{A} \times \mathcal{A} \setminus \Delta \subset \mathcal{D}$. Lemma 3.4 and Corollary 1.2 yield that $\mathcal{A}$ is the desired subset of $[\mathbb{N}]$. $\square$

We shall next pass to the proof of Theorem 1.1. We first prove some necessary lemmas.

Lemma 3.8. *Let $G \in [\mathbb{N}]^{<\infty}$ and $\xi < \omega$. The following are equivalent:*

1. *G is a member (resp. maximal member) of S_ξ.*
2. *For every $0 \leq \zeta \leq \xi$ there exist $n \in \mathbb{N}$ and $G_1 < \cdots < G_n$ successive members (resp. maximal members) of S_ζ so that $G = \cup_{i=1}^n G_i$ and $\{\min G_i : i \leq n\}$ is a member (resp. maximal member) of $S_{\xi-\zeta}$.*
3. *There exist $0 \leq \zeta \leq \xi$, $n \in \mathbb{N}$ and $G_1 < \cdots < G_n$ successive members (resp. maximal members) of S_ζ so that $G = \cup_{i=1}^n G_i$ and $\{\min G_i : i \leq n\}$ is a member (resp. maximal member) of $S_{\xi-\zeta}$.*

Proof. We show that all three conditions are equivalent for the members of S_ξ.

1. $\Rightarrow$ 2. By induction on ξ. If $\xi = 0$ the assertion is trivial. Suppose now $\xi \geq 1$ and that the assertion holds for $\xi - 1$. Let $\zeta \leq \xi$. If $\zeta = \xi$,

the assertion is again trivial. So assume $\zeta < \xi$. Choose $H_1 < \cdots < H_p$ in $S_{\xi-1}$ so that $p \leq \min H_1$ and $G = \cup_{i=1}^p H_i$. The induction hypothesis yields that for each $i \leq p$ there exist $H_{i1} < \cdots < H_{ir_i}$ in S_ζ so that $\{\min H_{ij} : j \leq r_i\}$ belongs to $S_{\xi-\zeta-1}$ and $H_i = \cup_{j=1}^{r_i} H_{ij}$. Let $\{G_1, \dots, G_n\}$ be an enumeration of the set $\{H_{ij} : j \leq r_i, i \leq p\}$ so that $G_1 < \cdots < G_n$. Note that $\{\min G_i : i \leq n\} = \cup_{i=1}^p \{\min H_{ij} : j \leq r_i\}$ and so it is a member of $S_{\xi-\zeta}$ as $p \leq \min H_{11} = \min H_1$.

2. $\Rightarrow$ 3. This implication is trivial.

3. $\Rightarrow$ 1. By induction on ξ. If $\xi = 0$ the assertion is trivial. Suppose now $\xi \geq 1$ and that the assertion holds for $\xi - 1$. Let $\zeta \leq \xi$. If $\zeta = \xi$, the assertion is again trivial. So assume $\zeta < \xi$. We first apply 1. $\Rightarrow$ 2. for the set $\{\min G_i : i \leq n\} \in S_{\xi-\zeta}$ to obtain $H_1 < \cdots < H_p$ in $S_{\xi-\zeta-1}$ so that $\{\min G_i : i \leq n\} = \cup_{i=1}^p H_i$ and $\{\min H_i : i \leq p\} \in S_1$. Set $I_i = \{j \leq n : \min G_j \in H_i\}$, $i \leq p$. Then $\cup_{j \in I_i} G_j \in S_{\xi-1}$, by the induction hypothesis since $\{\min G_j : j \in I_i\} = H_i$ which belongs to $S_{\xi-\zeta-1}$. Finally, $G = \cup_{i=1}^p (\cup_{j \in I_i} G_j) \in S_\xi$, as $\min G = \min H_1 \geq p$. The latter inequality holds because $\{\min H_i : i \leq p\} \in S_1$.

The proof for the case of maximal Schreier sets requires only minor modifications. Namely, all the sets which belong to an appropriate class S_α, $\alpha \leq \xi$ and appear in the previous arguments, can be taken to be maximal members of S_α. $\square$

Lemma 3.9. *Let $\xi < \omega_1$. Suppose that $L = (l_i)$, $M = (m_i)$ belong to $[\mathbb{N}]$ and satisfy $l_i < m_i < l_{i+1}$, for every $i \in \mathbb{N}$. Then $(e_{l_i}^\xi)$ is 2-equivalent to $(e_{m_i}^\xi)$. That is, $\| \sum_{i=1}^n a_i e_{l_i}^\xi \| \leq \| \sum_{i=1}^n a_i e_{m_i}^\xi \| \leq 2 \| \sum_{i=1}^n a_i e_{l_i}^\xi \|$, for every $n \in \mathbb{N}$ and all scalar sequences $(a_i)_{i=1}^n$.*

We omit the easy proof and pass to

Lemma 3.10. *Let $\xi < \omega$ and $0 \leq \zeta \leq \xi$. Then for every $L \in [\mathbb{N}]$, (ζ_n^L), considered as a sequence in X^ξ, is $12(\zeta+1)$-equivalent to $(e_{q_n}^{\xi-\zeta})$. Here we have set $q_n = \min F_n^\zeta(L)$, for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and $(a_i)_{i=1}^n$ be scalars. Choose $G \subset \{q_1, \dots, q_n\}$ with $G \in S_{\xi-\zeta}$ such that $\sum_{i \in I} |a_i| = \| \sum_{i=1}^n a_i e_{q_i}^{\xi-\zeta} \|$, where we have set $I = \{i \leq n : q_i \in G\}$. It follows that $H = \cup_{i \in I} F_i^\zeta(L) \in S_\xi$, by Lemma 3.8. Hence, $\| \sum_{i=1}^n a_i \zeta_i^L \|_\xi \geq \sum_{i \in I} |a_i|$ and thus $\| \sum_{i=1}^n a_i e_{q_i}^{\xi-\zeta} \| \leq \| \sum_{i=1}^n a_i \zeta_i^L \|_\xi$.

We next show that $\| \sum_{i=1}^n a_i \zeta_i^L \|_\xi \leq 12(\zeta+1) \| \sum_{i=1}^n a_i e_{q_i}^{\xi-\zeta} \|$. Let $G \in S_\xi[L]$ be maximal and put $G_i = F_i^\zeta(L) \cap G$, for all $i \leq n$. We apply Lemma 3.8 to find $p \in \mathbb{N}$ and $H_1 < \cdots < H_p$ maximal members of S_ζ so that $G = \cup_{j=1}^p H_j$ and $\{\min H_j : j \leq p\}$ is a maximal member of $S_{\xi-\zeta}$.

We claim that each of the G_i's can intersect at most two of the H_j's. Indeed, assume that for some i and $j_1 < j_2 < j_3$ we had that $G_i \cap H_{j_r} \neq \emptyset$, for all $r \leq 3$. Then $H_{j_2} \subset G_i$ because $H_{j_2} \subset [\min G_i, \max G_i]$. Thus,

$H_{j_2} \subset F_i^{\zeta}(L)$ and hence $H_{j_2} = F_i^{\zeta}(L)$, by the maximality of H_{j_2}. It follows that $H_{j_2} = G_i$ which is a contradiction as $H_{j_1} \cap H_{j_2} = \emptyset$.

Therefore our claim holds and evidently, for each $i \leq n$, G_i intersects either exactly one of the H_j's, or exactly two (consecutive) H_j's. We can thus partition $\{1, \dots, n\}$ in the following two subsets:

$$I_1 = \{i \leq n : G_i \subset H_j, \text{ for some } j \leq p\},$$

$$I_2 = \{i \leq n : \exists j_1 < j_2 \leq p, G_i \subset H_{j_1} \cup H_{j_2}, G_i \cap H_{j_r} \neq \emptyset, r \leq 2\}.$$

Let $k_i = \max F_i^{\zeta}(L)$, for all $i \leq n$. We now have the following

CLAIM. Suppose that $T_i \subset F_i^{\zeta}(L)$, for all $i \leq n$. Assume also that for each $i \leq n$ there exists $j \leq p$ so that $T_i \subset H_j$. Then

$$\Big|\sum_{i=1}^{n} a_i \zeta_i^L(\cup_{m=1}^{n} T_m)\Big| \leq (\zeta + 1)\Big\|\sum_{i=1}^{n} a_i e_{k_i}^{\xi - \zeta}\Big\|.$$

Once the claim is established we finish the proof as follows: Observe that our claim yields

$$\Big|\sum_{i \in I_1} a_i \zeta_i^L(G)\Big| \leq (\zeta + 1)\Big\|\sum_{i=1}^{n} a_i e_{k_i}^{\xi - \zeta}\Big\|.$$

On the other hand, if $i \in I_2$ there exist $A_i < B_i$ so that $G_i = A_i \cup B_i$ and each element of $\{A_i, B_i\}$ is contained in some H_j. Our claim then yields that

$$\Big|\sum_{i \in I_2} a_i \zeta_i^L(G)\Big| \leq 2(\zeta + 1)\Big\|\sum_{i=1}^{n} a_i e_{k_i}^{\xi - \zeta}\Big\|.$$

Therefore, $|\sum_{i=1}^{n} a_i \zeta_i^L(G)| \leq 3(\zeta + 1)\| \sum_{i=1}^{n} a_i e_{k_i}^{\xi - \zeta}\|$. It follows now, since G was arbitrary, that $\| \sum_{i=1}^{n} a_i \zeta_i^L\|_{\xi} \leq 6(\zeta + 1)\| \sum_{i=1}^{n} a_i e_{k_i}^{\xi - \zeta}\|$. The desired estimate follows now from Lemma 3.9.

We proceed now to prove our claim. Let $R_j = \{i \le n : T_i \ne \emptyset, T_i \subset H_j\}$, $j \le p$, and choose $i_j \in R_j$ such that $\max_{r \in R_j} |a_r| = |a_{i_j}|$.

$$
\begin{aligned}
\Big|\sum_{i=1}^n a_i \zeta_i^L(\cup_{m=1}^n T_m)\Big| &= \Big|\sum_{i=1}^n a_i \zeta_i^L(T_i)\Big| = \Big|\sum_{j=1}^p \sum_{i\in R_j} a_i \zeta_i^L(T_i)\Big| \\
&\le \sum_{j=1}^p \Big|\Big(\sum_{i\in R_j} a_i \zeta_i^L\Big)\big(\cup_{m\in R_j} T_m\big)\Big| \le \sum_{j=1}^p \Big\|\sum_{i\in R_j} a_i \zeta_i^L\Big\|_\zeta, \\
&\text{because } \cup_{m\in R_j} T_m \in S_\zeta, \\
&\le \sum_{j=1}^p (\zeta+1)\max_{i\in R_j}|a_i|, \text{ by Lemma 2.3 }, \\
&= \sum_{j=1}^p (\zeta+1)|a_{i_j}| \le (\zeta+1)\Big\|\sum_{i=1}^n a_i e_{k_i}^{\xi-\zeta}\Big\|.
\end{aligned}
$$

The last inequality holds since $T_{i_j} \subset H_j$ implies that $\min H_j \le k_{i_j}$, for all $j \le p$ and thus $\{k_{i_j} : j \le p\} \in S_{\xi-\zeta}$. The proof of the lemma is now complete. $\square$

We recall here that a sequence (x_n) in some Banach space is said to be an ℓ_1^ξ-*spreading model*, $\xi < \omega_1$, provided that there exists a constant $C > 0$ so that $\|\sum_{i\in F} a_i x_i\| \ge C \sum_{i\in F} |a_i|$, for every $F \in S_\xi$ and all scalars $(a_i)_{i\in F}$.

Remark . *It is easy to see that every subsequence of (e_n^ξ) is an ℓ_1^ξ-spreading model in X^ξ. However, Lemma 2.3 implies that no subsequence of (e_n^ξ) is an $\ell_1^{\xi+1}$-spreading model in X^ξ.*

Proposition 3.11. *Suppose $L = (l_i)$, $M = (m_i)$ belong to $[\mathbb{N}]$ and that $\xi < \omega$. Assume further that there exist a map $\psi\colon L \to M$ and a bounded linear operator $T\colon X_L^\xi \to X_M^\xi$ so that $T(e_l^\xi) = e_{\psi(l)}^\xi$, for every $l \in L$. Then there exists an integer $D > 0$ so that $\tau_\zeta(\psi^{-1}F) \le D$, for every $F \in S_\zeta[M]$ and all $0 \le \zeta \le \xi$.*

Proof. We first note that $\psi^{-1}F \in [L]^{<\infty}$, for every $F \in [M]^{<\infty}$. Indeed, if that were not the case, we would find $m \in M$ and $N \in [L]$ so that $\psi(l) = m$, for every $l \in N$. It follows that $T(\xi_i^N) = e_m^\xi$, for all $i \in \mathbb{N}$. But this contradicts Lemma 2.3 because T is bounded.

Fix $0 \le \zeta \le \xi$. Our first task is to show that $\sup_n \tau_\zeta\big(\psi^{-1}F_n^\zeta(P)\big) < \infty$, for every $P \in [M]$. Suppose this is not the case and so $\sup_n \tau_\zeta\big(\psi^{-1}F_n^\zeta(P)\big) = \infty$, for some $P \in [M]$. We claim that there exist a sequence of positive integers, (n_i), and a sequence of successive maximal $S_{\zeta+1}[L]$-sets, (G_i), so that letting $q_i = \min G_i$, for all $i \in \mathbb{N}$, the following is satisfied:

$$G_i \setminus \{q_i\} \subset \psi^{-1}F_{n_i}^\zeta(P), \text{ for all } i \in \mathbb{N}.$$

Indeed, choose n_1 so that $\tau_\zeta\big(\psi^{-1}F^\zeta_{n_1}(P)\big) > l_1$. Put $q_1 = l_1$. Because $\psi^{-1}F^\zeta_{n_1}(P)$ contains at least l_1 successive maximal $S_\zeta[L]$-sets, it is clear that there exists $H_1 \subset \psi^{-1}F^\zeta_{n_1}(P)$, $q_1 < \min H_1$, so that $G_1 = \{q_1\} \cup H_1$ is a maximal $S_{\zeta+1}[L]$-set.

Put $l_{t_1} = \max G_1$ and $w_1 = \tau_\zeta(\{l_1, \dots, l_{t_1}\})$. We can find $n_2 > n_1$ so that $\tau_\zeta\big(\psi^{-1}F^\zeta_{n_2}(P)\big) > w_1 + l_{t_1+1}$. Now, $\{l \in L : l \geq l_{t_1+1}\} \cap \psi^{-1}F^\zeta_{n_2}(P)$ must contain at least l_{t_1+1} successive maximal $S_\zeta[L]$-sets. If not, then $\tau_\zeta[\{l \in L : l \geq l_{t_1+1}\} \cap \psi^{-1}F^\zeta_{n_2}(P)] \leq l_{t_1+1}$ and thus $\tau_\zeta(\psi^{-1}F^\zeta_{n_2}(P)) \leq l_{t_1+1} + w_1$, by Lemma 3.2. But this contradicts the choice of n_2.

We set $q_2 = l_{t_1+1}$ and arguing as we did in the case $i = 1$, we can find $H_2 \subset \psi^{-1}F^\zeta_{n_2}(P)$, $q_2 < \min H_2$, so that $G_2 = \{q_2\} \cup H_2$ is a maximal $S_{\zeta+1}[L]$-set. We next put $l_{t_2} = \max G_2$ and continue in the same fashion to obtain sequences (n_i), (G_i) satisfying the desired properties.

Let $Q = \cup_{i=1}^\infty G_i$. Clearly, $Q \in [L]$ and $F^{\zeta+1}_i(Q) = G_i$, for all $i \in \mathbb{N}$. We now set $R = \cup_{i=1}^\infty F^\zeta_{n_i}(P)$. Then, $R \in [M]$ and $F^\zeta_i(R) = F^\zeta_{n_i}(P)$, for all $i \in \mathbb{N}$. We observe that if $q \in G_i \setminus \{q_i\}$, then $T(e^\xi_q) = e^\xi_m$, for some $m \in F^\zeta_i(R)$.

Next write $(\zeta+1)^Q_i = a_i e^\xi_{q_i} + (1-a_i)u_i$, for all $i \in \mathbb{N}$. Here, u_i is a convex combination of the vectors $(e^\xi_q)_{q \in G_i \setminus \{q_i\}}$ and $0 < a_i < 1$. Evidently, $\lim_i a_i = 0$. Observe that Tu_i is a convex combination of the vectors $(e^\xi_m)_{m \in F^\zeta_i(R)}$ and thus $\|Tu_i\|_\xi = 1$, for all $i \in \mathbb{N}$.

It must be the case that $\zeta < \xi$ for if not, Lemma 2.3 yields $\lim_i \|u_i\|_\xi = 0$. On the other hand $\|Tu_i\|_\xi = 1$, for all $i \in \mathbb{N}$. Hence T is not bounded contrary to our assumption. Therefore, $\zeta < \xi$ and so $\|u_i\|_\xi = 1$, for all $i \in \mathbb{N}$.

Recall that $\lim_i \|(1-a_i)Tu_i\|_\xi = 1$ and $(1-a_i)Tu_i$ is supported by $F^\zeta_i(R)$. Using Lemma 3.8, it is easy to check that $((1-a_i)Tu_i)$ is an $\ell_1^{\xi-\zeta}$-spreading model in X^ξ_M, and consequently, since $\lim_i a_i = 0$, $(T[(\zeta+1)^Q_i])$ is also an $\ell_1^{\xi-\zeta}$-spreading model in X^ξ_M. We conclude, as T is bounded, that $((\zeta+1)^Q_i)$ is an $\ell_1^{\xi-\zeta}$-spreading model in X^ξ. However, if we apply Lemma 3.10 we obtain that $(e^{\xi-\zeta-1}_{q_n})$ is an $\ell_1^{\xi-\zeta}$-spreading model in $X^{\xi-\zeta-1}$. But this contradicts with the remark after Lemma 3.10. Hence, $\sup_n \tau_\zeta\big(\psi^{-1}F^\zeta_n(P)\big) < \infty$, for every $P \in [M]$. It follows that

$$[M] = \cup_{k=1}^\infty \{P \in [M] : \tau_\zeta\big(\psi^{-1}F^\zeta_n(P)\big) \leq k, \forall n \in \mathbb{N}\}.$$

It is easily seen that every set in the union is closed in $[M]$. Baire's theorem now yields $k_\zeta \in \mathbb{N}$ and $r^\zeta_1 < \dots < r^\zeta_{s_\zeta}$ in M so that if $P \in [M]$, $P = (p_i)$, and $p_i = r^\zeta_i$, $i \leq s_\zeta$, then $\tau_\zeta\big(\psi^{-1}F^\zeta_n(P)\big) \leq k_\zeta$, for all $n \in \mathbb{N}$. It follows now that there exists $m^\zeta_0 \in M$ so that if $F \in S_\zeta[M]$, $\min F > m^\zeta_0$, then $\tau_\zeta(\psi^{-1}F) \leq k_\zeta$.

Finally, choose $D_\zeta \in \mathbb{N}$ so that $\tau_\zeta(\psi^{-1}F) \leq D_\zeta$, for every $F \in S_\zeta[M]$, $\max F \leq m^\zeta_0$. Part 5. of Lemma 3.2 now yields that $\tau_\zeta(\psi^{-1}F) \leq (D_\zeta +$

$k_\zeta)(\zeta+1)+1$, for every $F \in S_\zeta[M]$. To complete the proof we need only take $D = \max\{(D_\zeta + k_\zeta)(\zeta+1)+1 : \zeta \leq \xi\}$. □

Proposition 3.12. *Suppose $L = (l_i)$, $M = (m_i)$ belong to $[\mathbb{N}]$ and that $\xi < \omega$. Assume further that there exist a map $\psi \colon L \to M$ and an integer $D > 0$ so that $\tau_\zeta(\psi^{-1}F) \leq D$, for every $F \in S_\zeta[M]$ and all $0 \leq \zeta \leq \xi$. Then there exist integer constants E_ζ, $0 \leq \zeta \leq \xi$, so that $\tau_\zeta(\phi^{-1}F) \leq E_\zeta$, for every $F \in S_\zeta[M]$ and all $0 \leq \zeta \leq \xi$. Here, $\phi \colon L \to M$, is the natural bijection $\phi(l_i) = m_i$.*

Proof. If $\zeta = 0$ the assertion is trivial ($E_0 = 1$). Suppose the assertion holds for some $\zeta \leq \xi - 1$. We will show that $E_{\zeta+1} = [(\zeta+1)E_\zeta+1][(2D+1)(\zeta+2)+1]$ works for $\zeta+1$. Let $F \in S_{\zeta+1}[M]$, $F = \{m_{i_1}, \dots, m_{i_p}\}$. Our hypothesis yields that $\{l_{i_1}, \dots, l_{i_p}\}$ is contained in the union of $E_\zeta m_{i_1}$ $S_\zeta[L]$-sets and so $\tau_\zeta(\{l_{i_1}, \dots, l_{i_p}\}) \leq (\zeta+1)E_\zeta m_{i_1} + 1$ by part 5. of Lemma 3.2. Choose $q_1 \in \mathbb{N}$ so that the set $\{l_{i_1+j} : 0 \leq j \leq q_1\}$ is the union of exactly $[(\zeta+1)E_\zeta+1]m_{i_1}$ successive maximal $S_\zeta[L]$-sets.

CLAIM. $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq q_1\}) \leq E_{\zeta+1}$.

Once our claim is proven, we apply Lemmas 2.2, and 3.2 (parts 4. and 1.) to conclude that $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq p-1\}) \leq E_{\zeta+1}$ and $\tau_{\zeta+1}(\{l_{i_1}, \dots, l_{i_p}\}) \leq E_{\zeta+1}$.

To prove the claim we choose $q < q_1$ so that the set $\{l_{i_1+j} : 0 \leq j \leq q\}$ is the union of exactly m_{i_1} successive, maximal $S_\zeta[L]$-sets. Our task now is to show that $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq q\}) \leq (2D+1)(\zeta+2)+1$. The claim will then follow by applying parts 4. and 2. of Lemma 3.2.

We first observe that if $0 \leq j_0 \leq q$ is chosen so that $l_{i_1+j} \leq m_{i_1+j}$, for all $j \leq j_0$, then $\{l_{i_1+j} : 0 \leq j \leq j_0\}$ is contained in the union of $2D$ $S_{\zeta+1}[L]$-sets.

Indeed, $\{m_{i_1+j} : 0 \leq j \leq j_0\}$ belongs to $S_{\zeta+1}[M]$, by part 3. of Lemma 3.2 and the fact that $\tau_\zeta(\{l_{i_1+j} : 0 \leq j \leq j_0\} \leq m_{i_1}$. It follows now that $\Psi = \{\psi(l_{i_1+j}) : 0 \leq j \leq j_0, \psi(l_{i_1+j}) \geq m_{i_1}\}$ belongs to $S_{\zeta+1}[M]$. To see this let $\{m_{t_0} < \dots < m_{t_k}\}$, where $k \leq j_0$ and $i_1 \leq t_0$, be an enumeration of Ψ. Then $m_{t_j} \geq m_{i_1+j}$, for every $0 \leq j \leq k$. Since $\{m_{i_1+j} : 0 \leq j \leq k\}$ belongs to $S_{\zeta+1}[M]$ which is spreading, we conclude that Ψ belongs to $S_{\zeta+1}[M]$. Our hypothesis (for $\zeta+1$) yields that $\psi^{-1}(\Psi) = \{l_{i_1+j} : 0 \leq j \leq j_0, \psi(l_{i_1+j}) \geq m_{i_1}\}$ is contained in the union of D $S_{\zeta+1}[L]$-sets. On the other hand, the cardinality of the set $\{\psi(l_{i_1+j}) : 0 \leq j \leq j_0, \psi(l_{i_1+j}) < m_{i_1}\}$ is at most $i_1 - 1$. Our hypothesis (for $\zeta = 0$) now yields that the cardinality of the set $\{l_{i_1+j} : \psi(l_{i_1+j}) < m_{i_1}, 0 \leq j \leq j_0\}$ is at most $D(i_1-1)$. We deduce, since $l_{i_1} > i_1 - 1$, that $\{l_{i_1+j} : \psi(l_{i_1+j}) < m_{i_1}, 0 \leq j \leq j_0\}$ is contained in the union of D $S_1[L]$-sets. Hence, $\{l_{i_1+j} : 0 \leq j \leq j_0\}$ is contained in the union of $2D$ $S_{\zeta+1}[L]$-sets.

Next set $j_1 = \min\{j : 0 \leq j \leq q \text{ and } l_{i_1+j} > m_{i_1+j}\}$. If j_1 does not exist, then $l_{i_1+j} \leq m_{i_1+j}$, for all $0 \leq j \leq q$. We obtain, by our previous observation for $j_0 = q$, that $\{l_{i_1+j} : 0 \leq j \leq q\}$ is contained in the union of $2D$ $S_{\zeta+1}[L]$-sets.

If j_1 does exist, then $\{l_{i_1+j} : 0 \leq j < j_1\}$ is contained in the union of $2D$ $S_{\zeta+1}[L]$-sets. Indeed, this is obvious if $j_1 = 0$. If $j_1 \geq 1$ the assertion follows from our previous observation by taking $j_0 = j_1 - 1$. Finally, $\{l_{i_1+j} : j_1 \leq j \leq q\}$ belongs to $S_{\zeta+1}[L]$, since $l_{i_1+j_1} > m_{i_1+j_1} \geq m_{i_1}$ and $\{l_{i_1+j} : j_1 \leq j \leq q\}$ is contained in the union of m_{i_1} successive maximal $S_\zeta[L]$-sets. Thus, $\{l_{i_1+j} : 0 \leq j \leq q\}$ is contained in the union of $2D+1$ $S_{\zeta+1}[L]$-sets.

Concluding, in any case, the set $\{l_{i_1+j} : 0 \leq j \leq q\}$ is contained in the union of $2D+1$ $S_{\zeta+1}[L]$-sets. Hence, applying part 5. of Lemma 3.2, we obtain that $\tau_{\zeta+1}(\{l_{i_1+j} : 0 \leq j \leq q\}) \leq (2D+1)(\zeta+2)+1$, as desired. The proof of the proposition is complete. $\square$

Proposition 3.13. *Let $\xi < \omega_1$ and $L = (l_i)$, $M = (m_i)$ be in $[\mathbb{N}]$. Suppose that there exist $\delta > 0$ and a bounded linear operator $T \colon X_L^\xi \to X_M^\xi$ such that $\|T(e_l^\xi)\|_0 > \delta$, for every $l \in L$. Then there exist a map $\psi \colon L \to M$ and a bounded linear operator $R \colon X_L^\xi \to X_M^\xi$ such that $R(e_l^\xi) = e_{\psi(l)}^\xi$, for every $l \in L$.*

Proof. Following [11], given two infinite matrices (a_{ij}) and (d_{ij}), we shall call (d_{ij}) a block diagonal of (a_{ij}), if there exist (r_k), (s_k), increasing sequences of positive integers so that $d_{ij} = \begin{cases} a_{ij}, & \text{if } (i,j) \in \cup_{k=1}^\infty [r_k, r_{k+1}) \times [s_k, s_{k+1}) \\ 0, & \text{otherwise.} \end{cases}$
We can represent T as an infinite matrix (a_{ij}). Then $T(e_{l_i}^\xi) = \sum_{j=1}^\infty a_{ij} e_{m_j}^\xi$, for every $i \in \mathbb{N}$. Because $\|T(e_{l_i}^\xi)\|_0 > \delta$, for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $|a_{ij}| > \delta$. We can thus define a map $\psi \colon L \to M$ so that if $\psi(l_i) = m_j$, then $|a_{ij}| > \delta$. Observe that $\psi^{-1}\{m_j\}$ is finite, for all $j \in \mathbb{N}$, since $(T(e_{l_i}^\xi))$ is weakly null in X_M^ξ. In particular, $\psi(L) \in [M]$. Let $(m_{k_j})_{j=1}^\infty$ be the increasing enumeration of $\psi(L)$. Given $x = \sum_{i=1}^\infty \lambda_i e_{l_i} \in X_L^\xi$, we set $S(x) = \sum_{j=1}^\infty (\sum_{i=1}^\infty \lambda_i a_{ik_j}) e_{m_{k_j}}^\xi$. It follows, since T is bounded and (e_n^ξ) is unconditional, that S is a well defined bounded linear operator from X_L^ξ into $X_{\psi(L)}^\xi$. Moreover, the matrix representation (c_{ij}) of S with respect to the bases $(e_{l_i}^\xi)$ and $(e_{m_{k_j}}^\xi)$ is given by $c_{ij} = a_{ik_j}$, for all positive integers i, j.

We next consider the matrix (b_{ij}) given by

$$b_{ij} = \begin{cases} c_{ij}, & \text{if } \psi(l_i) = m_{k_j} \\ 0, & \text{otherwise.} \end{cases}$$

Note that there exists a unique non-zero entry in every row of the matrix (b_{ij}), while each column contains only finitely many non-zero entries. We can thus find p, a permutation of $\mathbb{N}$, so that the matrix $(b_{p(i)j})$ is a block diagonal of $(c_{p(i)j})$. Since $(c_{p(i)j})$ represents the bounded linear operator $S \colon X_L^\xi \to X_{\psi(L)}^\xi$ with respect to the bases $(e_{l_{p(i)}}^\xi)$ and $(e_{m_{k_j}}^\xi)$, and $(b_{p(i)j})$ is a block diagonal of $(c_{p(i)j})$, Proposition 1.c.8 of [11] yields that $(b_{p(i)j})$

also represents a bounded linear operator from X_L^ξ into $X_{\psi(L)}^\xi$ with respect to the bases $(e_{l_{p(i)}}^\xi)$ and $(e_{m_{k_j}}^\xi)$. Consequently, (b_{ij}) represents a bounded linear operator $W\colon X_L^\xi \to X_{\psi(L)}^\xi$ with respect to the bases $(e_{l_i}^\xi)$ and $(e_{m_{k_j}}^\xi)$ which evidently satisfies $W(e_{l_i}^\xi) = a_{ik_j} e_{\psi(l_i)}^\xi$ (where $\psi(l_i) = m_{k_j}$), for all $i \in \mathbb{N}$. Because $|a_{ik_j}| > \delta$, if $\psi(l_i) = m_{k_j}$, and (e_n^ξ) is unconditional, we obtain that there exists a bounded linear operator $R\colon X_L^\xi \to X_M^\xi$ such that $R(e_l^\xi) = e_{\psi(l)}^\xi$, for every $l \in L$. □

We are now ready for the

Proof of Theorem 1.1. 3. ⇒ 1. and 2. ⇒ 3. are immediate. To prove that 1. implies 2. we first apply Proposition 3.13 to obtain a map $\psi\colon L \to M$ and a bounded linear operator $R\colon X_L^\xi \to X_M^\xi$ such that $R(e_l^\xi) = e_{\psi(l)}^\xi$, for every $l \in L$. Propositions 3.11 and 3.12 will then yield a constant $E > 0$ such that $\tau_\zeta(\phi^{-1}F) \leq E$, for every $F \in S_\zeta[M]$ and $0 \leq \zeta \leq \xi$. (Where $\phi\colon L \to M$ is the natural bijection.) The result now follows from Lemma 3.4. □

To obtain Corollary 1.2 we shall need the following

Lemma 3.14. *Let $\xi < \omega$ and $s = (u_n)$ be a bounded block basis of (e_n^ξ) such that $\lim_n \|u_n\|_0 = 0$. Then for every $N \in [\mathbb{N}]$ and $0 \leq \zeta \leq \xi$ there exists $M \in [N]$ so that $\lim_n \|\zeta_n^M \cdot s\|_\zeta = 0$. (Given $\mu = \sum_{i=1}^\infty a_i e_i \in c_{00}$, we denote by $\mu \cdot s$ the vector $\sum_{i=1}^\infty a_i u_i$ which of course belongs to c_{00}.)*

Proof. If $\zeta = 0$, the assertion follows from the fact that $\lim_n \|u_n\|_0 = 0$. Assume now that $\zeta \leq \xi - 1$ and that the assertion holds for ζ. Let $N \in [\mathbb{N}]$ and $\epsilon > 0$. We will find $Q \in [N]$ so that $\|(\zeta+1)_1^Q \cdot s\|_{\zeta+1} < \epsilon$. Once this is accomplished, we can choose $(Q_i) \subset [N]$ so that $\|(\zeta+1)_1^{Q_i} \cdot s\|_{\zeta+1} < \epsilon_i$, where $\lim_i \epsilon_i = 0$ and $F_1^{\zeta+1}(Q_1) < F_1^{\zeta+1}(Q_2) < \cdots$. Letting $M = \cup_{i=1}^\infty F_1^{\zeta+1}(Q_i)$, we obtain that $(\zeta+1)_1^{Q_i} = (\zeta+1)_i^M$, for all $i \in \mathbb{N}$ and thus $\lim_i \|(\zeta+1)_i^M \cdot s\|_{\zeta+1} = 0$.

We now pass to the construction of Q. By the induction hypothesis we can choose a sequence $(P_i) \subset [N]$ satisfying the following properties:

1. $F_1^\zeta(P_1) < F_1^\zeta(P_2) < \cdots$.
2. $\min F_1^\zeta(P_1) > \frac{2+2b}{\epsilon}$, where b is chosen so that $\|u_n\|_\xi \leq b$, for every $n \in \mathbb{N}$.
3. $\|\zeta_1^{P_i} \cdot s\|_\zeta < \frac{1}{2^i k_{i-1}}$, for all $i \geq 2$. Here we have set $k_i = \max supp(\zeta_1^{P_i} \cdot s)$.

Put $Q = \cup_{i=1}^\infty F_1^\zeta(P_i)$. We are going to show that $\|\sum_{i=1}^n \zeta_i^Q \cdot s\|_{\zeta+1} \leq 2 + 2b$, for every $n \in \mathbb{N}$. Note that $\zeta_i^Q = \zeta_1^{P_i}$ and $F_i^\zeta(Q) = F_1^\zeta(P_i)$, for all $i \in \mathbb{N}$.

Let $G \in S_{\zeta+1}$. Let also $\{i_1, \dots, i_p\}$ be an enumeration of $\{i \leq n : supp(\zeta_i^Q \cdot s) \cap G \neq \emptyset\}$. Choose $l \leq \min G$ and $G_1 < \cdots < G_l$ in S_ζ so that $G = \cup_{i=1}^l G_i$.

Then $|(\zeta^Q_{i_1} \cdot s)(G)| \le b$. Further,

$$\begin{aligned}\left|\sum_{t=2}^{p} \zeta^Q_{i_t} \cdot s(G)\right| &= \left|\sum_{j=1}^{l}\sum_{t=2}^{p}(\zeta^Q_{i_t} \cdot s)(G_j)\right| \le \sum_{j=1}^{l}\sum_{t=2}^{p}|(\zeta^Q_{i_t} \cdot s)(G_j)| \\ &\le \sum_{j=1}^{l}\sum_{t=2}^{p}\|\zeta^Q_{i_t} \cdot s\|_\zeta \le \sum_{j=1}^{l}\sum_{t=2}^{p}\frac{1}{2^{i_t}k_{i_t-1}} \\ &\le \frac{l}{k_{i_1}} \le 1, \text{ since } l \le \min G \le k_{i_1}.\end{aligned}$$

Therefore, $|\sum_{i=1}^{n}(\zeta^Q_i \cdot s)(G)| = |\sum_{t=1}^{p}(\zeta^Q_{i_t} \cdot s)(G)| \le b+1$. It follows that $\|\sum_{i=1}^{n}\zeta^Q_i \cdot s\|_{\zeta+1} \le 2+2b$. If we take $n = \min F_1^\zeta(P_1)$, we obtain that $\|(\zeta+1)^Q_1 \cdot s\|_{\zeta+1} < \epsilon$. The proof of the lemma is now complete. □

Proof of Corollary 1.2. Let $T\colon X^\xi_L \to X^\xi_M$ be an isomorphic embedding. We apply Theorem 1.1 to show that $(e^\xi_{l_n})$ dominates $(e^\xi_{m_n})$. Indeed, we need only check that $\inf_{l\in L}\|T(e^\xi_l)\|_0 > 0$. If that were not the case, let (x_i) be a subsequence of $(T(e^\xi_{l_i}))$ such that $\lim_i \|x_i\|_0 = 0$. By a standard perturbation result we can assume, without loss of generality, that for some block basis (u_i) of $(e^\xi_{l_i})$ and a null sequence of positive scalars (ϵ_i) we have that $\|x_i - u_i\|_\xi < \epsilon_i$, for all $i \in \mathbb{N}$. It follows that also $\lim_i \|u_i\|_0 = 0$, and thus Lemma 3.14 yields $N \in [\mathbb{N}]$ so that $\lim_i \|\xi^N_i \cdot s\|_\xi = 0$, where $s = (u_i)$. But then $\lim_i \|\xi^N_i \cdot x\|_\xi = 0$ as well. $(x = (x_i))$ This is a contradiction because (x_i) is equivalent to a subsequence of $(e^\xi_{l_i})$ and thus it is an ℓ^ξ_1-spreading model. Hence, $(e^\xi_{l_n})$ dominates $(e^\xi_{m_n})$ completing the proof of part 1. Parts 2 and 3 are immediate consequences of Theorem 1.1. □

We recall that a Banach space X is said to be *primary* if, for every bounded linear projection P on X, either PX or $(I-P)X$ is isomorphic to X.

Corollary 3.15. *X^ξ_N is not primary, for every $N \in [\mathbb{N}]$ and all $1 \le \xi < \omega$.*

Proof. We first let $\mathcal{F} = \{(L,M) \in [N]\times[N] : L \cup M = N,\ L \cap M = \emptyset\}$. $\mathcal{F}$ is easily seen to be closed in $[N]\times[N]$ and thus it is a Polish space. We next set $\mathcal{G} = \{(L,M) \in \mathcal{F} : d_\xi(N,L) = d_\xi(N,M) = \infty\}$. Arguing as we did in the proof of Lemma 3.5 we obtain that $\mathcal{G}$ is a G_δ dense subset of $\mathcal{F}$. If $(L,M) \in \mathcal{G}$ then $X^\xi_N = X^\xi_L \bigoplus X^\xi_M$. However, Theorem 1.1 implies that X^ξ_N is not isomorphic to a subspace of either X^ξ_L or X^ξ_M. □

4. Subspaces spanned by block bases

In this section we investigate subspaces of X^ξ spanned by block bases of (e^ξ_n). We first show that there exists a block basis of (e^ξ_n) spanning a complemented subspace of X^ξ which is not isomorphic to X^ζ_M, for every $M \in [\mathbb{N}]$ and all $0 \le \zeta \le \xi$.

Lemma 4.1. *Let $x_1 < \cdots < x_p$ be a finite block basis of (e_n), the unit vector basis of c_{00}. Let also $G_1 < \cdots < G_q$ be finite subsets of $\mathbb{N}$ and $(a_i)_{i=1}^p$ be scalars. Assume that there exists $C > 0$ such that $|(\sum_{i\in I} a_i x_i)(\cup_{j\in J} G_j)| \leq C$, whenever $I \subset \{1,\dots,p\}$ and $J \subset \{1,\dots,q\}$ satisfy one of the following two conditions:*

1. *$I = \cup_{j\in J} I_j$, $I_{j_1} < I_{j_2}$ if $j_1 < j_2$ and $I_j = \{i \in I : supp x_i \cap G_j \neq \emptyset\}$, for all $j \in J$.*
2. *$J = \cup_{i\in I} J_i$, $J_{i_1} < J_{i_2}$ if $i_1 < i_2$ and $J_i = \{j \in J : supp x_i \cap G_j \neq \emptyset\}$, for all $i \in I$.*

Then $|(\sum_{i=1}^p a_i x_i)(\cup_{j=1}^q G_j)| \leq 3C$.

Proof. Given $j \leq q$, we let $T_j = \{i \leq p : supp x_i \cap G_j \neq \emptyset\}$. We also let $J = \{j \leq q : T_j \neq \emptyset\}$ and $J_1 = \{j \in J : |T_j| = 1\}$. Set $J_2 = J \setminus J_1$. Given $j \in J_2$ we let $s_j = \min T_j$ and $t_j = \max T_j$. We observe that $s_{j_1} < t_{j_1} \leq s_{j_2}$, for every $j_1 < j_2$ in J_2.

Next, we define a map $\sigma\colon J_1 \to \{1,\dots,p\}$ so that $\{\sigma(j)\} = T_j$, for every $j \in J_1$. Note that $\sigma(J_1)$ and J_1 satisfy condition 2. and therefore

$$\Big|\big(\sum_{i=1}^p a_i x_i\big)\big(\cup_{j\in J_1} G_j\big)\Big| = \Big|\big(\sum_{i\in\sigma(J_1)} a_i x_i\big)\big(\cup_{j\in J_1} G_j\big)\Big| \leq C.$$

Suppose now that $J_2 = \{j_1,\dots,j_k\}$ and put $J_3 = \{j_r : r \leq k, r \text{ is odd }\}$ and $J_4 = \{j_r : r \leq k, r \text{ is even }\}$. It follows that $\cup_{j\in J_m} T_j$ and J_m, $m \in \{3,4\}$, satisfy condition 1. and thus

$$\Big|\big(\sum_{i=1}^p a_i x_i\big)\big(\cup_{j\in J_m} G_j\big)\Big| = \Big|\big(\sum_{i\in\cup_{j\in J_m} T_j} a_i x_i\big)\big(\cup_{j\in J_m} G_j\big)\Big| \leq C,\ m \in \{3,4\}.$$

Hence, $|(\sum_{i=1}^p a_i x_i)(\cup_{j\in J_2} G_j)| \leq 2C$. The assertion follows since

$$\big(\sum_{i=1}^p a_i x_i\big)\big(\cup_{j=1}^q G_j\big) = \big(\sum_{i=1}^p a_i x_i\big)\big(\cup_{j\in J_1} G_j\big) + \big(\sum_{i=1}^p a_i x_i\big)\big(\cup_{j\in J_2} G_j\big).$$

□

Lemma 4.2. *Let $1 \leq \zeta \leq \xi < \omega$ and (x_n) be a block basis of (e_n^ξ) so that for some $b > 0$, $\|x_n\|_\xi < b$, for every $n \in \mathbb{N}$. Let also $k_n = \max supp x_n$, for every $n \in \mathbb{N}$, and suppose that $\|x_n\|_{\zeta-1} < \frac{1}{2^{k_{n-1}}}$, for every $n \geq 2$. Then $|(\sum_{i=1}^n a_i x_i)(H)| \leq (2+b)\max_{i\leq n} |a_i|$, for every $H \in S_\zeta$, $n \in \mathbb{N}$ and all scalar sequences $(a_i)_{i=1}^n$.*

Proof. Let $H \in S_\zeta$ and put $i_0 = \min\{i \leq n : supp x_i \cap H \neq \emptyset\}$. We may write $H = \cup_{j=1}^r H_j$, where $r \leq \min H$ and $H_1 < \cdots < H_r$ belong to $S_{\zeta-1}$.

Note that $\min H \le k_{i_0}$. We also observe that $|x_i(H)| \le r\|x_i\|_{\zeta-1}$ and hence

$$\big|\sum_{i=i_0+1}^{n} a_i x_i(H)\big| \le \sum_{i=i_0+1}^{n} |a_i| r \|x_i\|_{\zeta-1}$$
$$\le r(\max_{i\le n}|a_i|)\sum_{i=i_0+1}^{\infty}\frac{1}{2^{k_{i-1}}} \le 2\max_{i\le n}|a_i|.$$

Finally, $|x_{i_0}(H)| \le \|x_{i_0}\|_\xi < b$ and thus
$|(\sum_{i=1}^n a_i x_i)(H)| \le (2+b)\max_{i\le n}|a_i|$, as desired. □

Our next proposition is a partial generalization of Lemma 3.10.

Proposition 4.3. *Let $\xi < \omega$ and (x_n) be a semi-normalized block basis of (e_n^ξ). Set $\zeta = \min\{\alpha \le \xi : \inf_n \|x_n\|_\alpha > 0\}$. Then there exists a subsequence of (x_n) which is equivalent to a subsequence of $(e_n^{\xi-\zeta})$.*

Proof. Choose $\delta > 0$, $b > 0$ so that $\delta < \|x_n\|_\zeta$ and $\|x_n\|_\xi < b$, for every $n \in \mathbb{N}$. Assume first that $\zeta \ge 1$. Then we choose inductively $n_1 < n_2 < \cdots$ so that $\|x_{n_i}\|_{\zeta-1} < \frac{1}{2^{k_{i-1}}}$, for every $i \ge 2$, where $k_i = \max \mathit{supp} x_{n_i}$. For every $i \in \mathbb{N}$ we can find $F_i \in S_\zeta$, $F_i \subset \mathit{supp} x_{n_i}$, so that $|x_{n_i}|(F_i) > \delta$. Put $m_i = \min F_i$. We are going to show that (x_{n_i}) is equivalent to $(e_{m_i}^{\xi-\zeta})$. To this end let $k \in \mathbb{N}$ and $(a_i)_{i=1}^k$ be scalars. We first show that $\|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\| \le \delta^{-1}\|\sum_{i=1}^k a_i x_{n_i}\|_\xi$. Indeed, if $G \subset \{m_1,\dots,m_k\}$ belongs to $S_{\xi-\zeta}$ then set $A = \{i \le k : m_i \in G\}$. We have the following estimate

$$\sum_{i\in A}|a_i| \le \delta^{-1}\sum_{i\in A}|a_i||x_{n_i}|(F_i) \le \delta^{-1}\|\sum_{i=1}^k a_i x_{n_i}\|_\xi$$

as $\cup_{i\in A}F_i \in S_\xi$, by Lemma 3.8.

Next, let $G \in S_\xi$. Lemma 3.8 yields $G_1 < \cdots < G_q$ in S_ζ with $\{\min G_j : j \le q\}$ belonging to $S_{\xi-\zeta}$ and so that $G = \cup_{j=1}^q G_j$. We shall apply Lemma 4.1 in order to estimate $|(\sum_{i=1}^k a_i x_{n_i})(\cup_{j=1}^q G_j)|$. Let $I \subset \{1,\dots,k\}$ and $J \subset \{1,\dots,q\}$ satisfy condition 1. of Lemma 4.1. Then $I_j = \{i \in I : \mathit{supp} x_{n_i} \cap G_j \ne \emptyset\}$, for every $j \in J$. We choose $i_j \in I_j$ such that $|a_{i_j}| = \max_{i\in I_j}|a_i|$, for every $j \in J$. Fix $j_0 \in J$.

$$\Big|\big(\sum_{i\in I_{j_0}} a_i x_{n_i}\big)\big(\cup_{j\in J}G_j\big)\Big| = \Big|\sum_{i\in I_{j_0}} a_i x_{n_i}(G_{j_0})\Big| \le (2+b)\max_{i\in I_{j_0}}|a_i|$$
$$= (2+b)|a_{i_{j_0}}|, \text{ by Lemma 4.2 .}$$

Hence $|(\sum_{i\in I} a_i x_{n_i})(\cup_{j\in J}G_j)| \le (2+b)\sum_{j\in J}|a_{i_j}|$.

Note also that $\{m_{i_j} : j \in J \setminus \{\min J\}\}$ belongs to $S_{\xi-\zeta}$. This is so since $\mathit{supp} x_{n_i} \cap G_j \ne \emptyset$, whenever $i \in I_j$ and $j \in J$, and thus $\min G_{j_1} < \min \mathit{supp} x_{n_i} \le m_i$, for every $i \in I_{j_2}$ and $j_1 < j_2$ in J. In particular,

$\min G_{j_1} < m_{i_{j_2}}$, when $j_1 < j_2$ in J. Since $S_{\xi-\zeta}$ is spreading we obtain that $\{m_{i_j} : j \in J \setminus \{\min J\}\}$ belongs to $S_{\xi-\zeta}$. It follows now that $\sum_{j\in J\setminus\{\min J\}} |a_{i_j}| \leq \| \sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\|$ and hence

$$\Big|\big(\sum_{i\in I} a_i x_{n_i}\big)\big(\cup_{j\in J} G_j\big)\Big| \leq 2(2+b)\Big\|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\Big\|.$$

We shall now assume that $I \subset \{1,\dots,k\}$ and $J \subset \{1,\dots,q\}$ satisfy condition 2. of Lemma 4.1. Then $J_i = \{j \in J : supp x_{n_i} \cap G_j \neq \emptyset\}$, for all $i \in I$. An argument similar to that in the preceding paragraph, yields that $\{m_i : i \in I \setminus \{\min I\}\}$ belongs to $S_{\xi-\zeta}$. It follows that $\sum_{i\in I} |a_i| \leq 2\| \sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\|$. Finally,

$$\begin{aligned}\Big|\big(\sum_{i\in I} a_i x_{n_i}\big)\big(\cup_{j\in J} G_j\big)\Big| &= \Big|\sum_{i\in I} a_i x_{n_i}\big(\cup_{j\in J_i} G_j\big)\Big| \\ &\leq b\sum_{i\in I}|a_i|, \text{ as } \cup_{j\in J_i} G_j \in S_\xi, \\ &\leq 2b\Big\|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\Big\|.\end{aligned}$$

We deduce from Lemma 4.1 that

$$\Big|\big(\sum_{i=1}^k a_i x_{n_i}\big)\big(\cup_{j=1}^q G_j\big)\Big| \leq 6(2+b)\Big\|\sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\Big\|,$$

and hence $\| \sum_{i=1}^k a_i x_{n_i}\|_\xi \leq 12(2+b)\| \sum_{i=1}^k a_i e_{m_i}^{\xi-\zeta}\|$.

To complete the proof we need to consider the case $\zeta = 0$. We now choose $m_n \in supp x_n$ such that $|x_n|(\{m_n\}) > \delta$, for all $n \in \mathbb{N}$. We are going to show that (x_n) is equivalent to $(e_{m_n}^\xi)$. Arguing as we did in the case $\zeta \geq 1$ we obtain that $\| \sum_{i=1}^k a_i e_{m_i}^\xi\| \leq \delta^{-1}\| \sum_{i=1}^k a_i x_i\|_\xi$, for every $k \in \mathbb{N}$ and all scalar sequences $(a_i)_{i=1}^k$.

Next let $G \in S_\xi$ and put $I = \{i \leq k : supp x_i \cap G \neq \emptyset\}$. Then

$$\begin{aligned}\Big|\sum_{i=1}^k a_i x_i(G)\Big| &\leq \sum_{i\in I}|a_i||x_i(G)| \leq b\sum_{i\in I}|a_i| \\ &\leq 2b\Big\|\sum_{i=1}^k a_i e_{m_i}^\xi\Big\|, \text{ as } \{m_i : i \in I\setminus\{\min I\}\} \in S_\xi.\end{aligned}$$

Hence $\| \sum_{i=1}^k a_i x_i\|_\xi \leq 4b\| \sum_{i=1}^k a_i e_{m_i}^\xi\|$. The proof of the proposition is now complete. □

As an immediate consequence of Proposition 4.3 we obtain

Corollary 4.4. *For every semi-normalized weakly null sequence in X^ξ, $\xi < \omega$, there exist $\zeta \leq \xi$ and a subsequence which is equivalent to a subsequence of (e_n^ζ).*

Lemma 4.5. *Let $1 \leq \xi < \omega$ and (F_n) be a sequence of successive members of S_ξ satisfying the following*

1. $\big(\tau_{\xi-1}(F_n)\big)$ *increases to ∞.*
2. $\sup_n \frac{\min F_n}{\tau_{\xi-1}(F_{n+k})} > k$, *for every $k \in \mathbb{N}$.*

Let (u_n) be a convex block basis of (e_n^ξ) such that $\mathrm{supp} u_n = F_n$, for every $n \in \mathbb{N}$. Assume further that $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$. Then the closed linear span of (u_n) in X^ξ is not isomorphic to X_M^ζ, for every $\zeta \leq \xi$ and $M \in [\mathbb{N}]$.

Proof. (u_n) is normalized in X^ξ since $F_n \in S_\xi$, for every $n \in \mathbb{N}$. We let X denote the closed linear span of (u_n) in X^ξ. Because $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$, we deduce from Proposition 4.3 that every semi-normalized block basis of (u_n) admits a subsequence equivalent to the unit vector basis of c_0. Indeed, let (v_n), $v_n = \sum_{i\in G_n} b_i u_i$, be a semi-normalized block basis of (u_n). Note that (b_n) is bounded since (v_n) is. But also, $\lim_n \sum_{i\in G_n} \|u_i\|_{\xi-1} = 0$, since $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$. Hence $\lim_n \|v_n\|_{\xi-1} = 0$ and therefore Proposition 4.3 (for $\zeta = \xi$) yields a subsequence of (v_n) equivalent to the unit vector basis of c_0.

It follows that every semi-normalized weakly null sequence in X admits a subsequence equivalent to the unit vector basis of c_0. That is, X has property (S) [7]. However, X_M^ζ fails property (S) when $\zeta \geq 1$ and $M \in [\mathbb{N}]$. Thus, X_M^ζ is not isomorphic to a subspace of X for every $1 \leq \zeta \leq \xi$ and $M \in [\mathbb{N}]$.

To complete the proof we show that X is not isomorphic to c_0. This is accomplished by showing that for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ so that $(u_{n+i})_{i=1}^k$ is isometrically equivalent to the unit vector basis of ℓ_1^k. In particular X contains uniformly complemented ℓ_1^k's. It is a well known fact that c_0 fails this property.

We let $k \in \mathbb{N}$ and choose according to 2. $n \in \mathbb{N}$ so that $\frac{\min F_n}{\tau_{\xi-1}(F_{n+k})} > k$. Condition 1. now yields that $\sum_{i=1}^k \tau_{\xi-1}(F_{n+i}) < \min F_n$ and thus $\cup_{i=1}^k F_{n+i} \in S_\xi$. Hence

$$\Big\|\sum_{i=1}^k a_i u_{n+i}\Big\|_\xi \geq \sum_{i=1}^k |a_i| u_{n+i}(F_{n+i}) = \sum_{i=1}^k |a_i|,$$

for every scalar sequence $(a_i)_{i=1}^k$. Therefore $(u_{n+i})_{i=1}^k$ is isometrically equivalent to the unit vector basis of ℓ_1^k. □

Proposition 4.6. *Let $1 \leq \xi < \omega$. There exists a normalized convex block basis (u_n) of (e_n^ξ) so that letting $F_n = \mathrm{supp} u_n$, for every $n \in \mathbb{N}$, the following are satisfied*

1. *$\tau_{\xi-1}(F_n) = n^2$ and $\min F_n > k(n+k)^2$, for every n and k in $\mathbb{N}$ such that $k < n$.*
2. *$X = [u_n : n \in \mathbb{N}]$ is not isomorphic to X^ζ_M for every $\zeta \leq \xi$ and $M \in [\mathbb{N}]$.*
3. *X is complemented in X^ξ.*

Proof. We inductively choose a sequence of integer intervals (F_n) such that for every $n \in \mathbb{N}$

$$\min F_n > \max\{k(n+k)^2 : k < n\} \cup \{\min F_{n-1}\} \text{ and } \tau_{\xi-1}(F_n) = n^2.$$

Put $M_n = F_n \cup \{m \in \mathbb{N} : m > \max F_n\}$, for every $n \in \mathbb{N}$. We then define

$$u_n = \frac{1}{n^2}\sum_{i=1}^{n^2} (\xi-1)_i^{M_n},\ n \in \mathbb{N}.$$

Condition 1. is an immediate consequence of the inductive construction. This condition implies that in fact $F_n \in S_\xi$, for every $n \in \mathbb{N}$ and thus (u_n) is indeed a normalized convex block basis of (e^ξ_n). We also obtain from Lemma 2.3 that $\|u_n\|_{\xi-1} \leq \frac{\xi}{n^2}$ and so $\sum_{n=1}^\infty \|u_n\|_{\xi-1} < \infty$. Hence condition 2. holds in view of Lemma 4.5. It remains to establish that X is complemented in X^ξ. To this end we define a map $P\colon c_{00} \to c_{00}$ by

$$P(x) = \sum_{i=1}^\infty x(F_i)u_i,\ \text{for all } x \in c_{00}.$$

Clearly P is well defined and linear. It is also clear that $P(u_i) = u_i$, for every $i \in \mathbb{N}$. Our objective is to show that P is bounded with respect to the $\|\cdot\|_\xi$-norm on c_{00}, for then P will extend to a bounded linear projection on X^ξ with range equal to X. To achieve our goal it suffices to show that if $G \in S_\xi$ is maximal, then $(\sum_{i=1}^p x(F_i)u_i)(G) \leq 18\xi$, for every $p \in \mathbb{N}$ and $x \in c_{00}$, $\|x\|_\xi \leq 1$, with $x(\{i\}) \geq 0$, $i \in \mathbb{N}$.

According to condition 1. of our hypothesis, for every $i \in \mathbb{N}$ there exist $F_{i1} < \cdots < F_{ii^2}$ successive $S_{\xi-1}$ sets so that $F_i = \cup_{k=1}^{i^2} F_{ik}$, and $\{\min F_{ik} : k \leq i^2\} \in S_1$. Next let $q = \min G$ and choose $G_1 < \cdots < G_q$ maximal members of $S_{\xi-1}$ so that $G = \cup_{j=1}^q G_j$. Of course $\{\min G_j : j \leq q\}$ is maximal in S_1. We shall apply Lemma 4.1. Let $I \subset \{1,\dots p\}$ and $J \subset \{1,\dots q\}$ satisfy condition 1. of Lemma 4.1. Recall that $I_j = \{i \in I : \mathit{supp} u_i \cap G_j \neq \emptyset\}$, $j \in J$. For each $j \in J$ we choose $i_j \in I_j$ and $k_j \leq i_j^2$ such

that $x(F_{i_j k_j}) = \max_{k \leq i^2, i \in I_j} x(F_{ik})$. We have the following estimate

$$\begin{aligned}
\left(\sum_{i\in I} x(F_i)u_i\right)\left(\cup_{j\in J} G_j\right) &= \sum_{j\in J}\sum_{i\in I_j} x(F_i)u_i(G_j) \\
&= \sum_{j\in J}\sum_{i\in I_j} x(F_i)\frac{1}{i^2}\sum_{k=1}^{i^2}(\xi-1)_k^{M_i}(G_j) \\
&\leq \sum_{j\in J}\sum_{i\in I_j} x(F_{i_j k_j})\sum_{k=1}^{i^2}(\xi-1)_k^{M_i}(G_j), \\
&\text{since } F_i = \cup_{k=1}^{i^2} F_{ik}, \\
&\leq \sum_{j\in J} x(F_{i_j k_j})\sum_{i\in I_j}\sum_{k=1}^{i^2}(\xi-1)_k^{M_i}(G_j) \\
&\leq \sum_{j\in J} x(F_{i_j k_j})\xi, \text{ by Lemma 2.3,} \\
&\leq \xi x(\cup_{j\in J} F_{i_j k_j}) \leq 2\xi.
\end{aligned}$$

The last inequality holds because $\|x\|_\xi \leq 1$ and $\cup_{j\in J\setminus\{\min J\}} F_{i_j k_j} \in S_\xi$. Indeed, $\min G_{j_1} < \min F_{i_{j_2} k_{j_2}}$ when $j_1 < j_2$ in J and therefore, as $\{\min G_j : j \leq q\} \in S_1$, $\cup_{j\in J\setminus\{\min J\}} F_{i_j k_j}$ belongs to S_ξ by Lemma 3.8.

Next assume that $I \subset \{1, \dots, p\}$ and $J \subset \{1, \dots, q\}$ satisfy condition 2. of Lemma 4.1. Then $J_i = \{j \in J : supp u_i \cap G_j \neq \emptyset\}$, $i \in I$. We set $H_i = \{j \in J_i : G_j \subset supp u_i\}$, $i \in I$. Since $supp u_i = F_i$ is an interval, $|J_i| \leq |H_i| + 2$, for all $i \in I$. Moreover, since each G_j is a maximal $S_{\xi-1}$ set and $\tau_{\xi-1}(F_i) = i^2$, we have that $|H_i| \leq i^2$, for all $i \in I$. To estimate $\sum_{i\in I}\sum_{j\in J_i\setminus H_i} x(F_i)u_i(G_j)$, choose $j_i \in J_i \setminus H_i$, for every $i \in I$ (we have assumed without loss of generality that $J_i \setminus H_i \neq \emptyset$). Then, the sets I and $\{j_i : i \in I\}$ satisfy condition 1. of Lemma 4.1. We deduce from our preceding work that

$$\sum_{i\in I}\sum_{j\in J_i\setminus H_i} x(F_i)u_i(G_j) \leq 4\xi,$$

as $|J_i \setminus H_i| \leq 2$, for every $i \in I$. We next choose, for every $i \in I$, $R_i \subset \{1, \dots, i^2\}$ with $|R_i| = |H_i|$ and such that

$$\frac{1}{i^2}\sum_{k=1}^{i^2} x(F_{ik}) \leq \frac{1}{|H_i|}\sum_{k\in R_i} x(F_{ik}).$$

This choice is possible since $|H_i| \leq i^2$. (We make use of the following fact: Let $(a_i)_{i=1}^n$ be scalars with $a_i \leq a_j$, $i \leq j$, and let $k < n$. Then

$\frac{1}{n}\sum_{i=1}^{n} a_i \leq \frac{1}{n-k}\sum_{i=k+1}^{n} a_i$.) We now have that

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in H_i} x(F_i)u_i(G_j) &= \sum_{i\in I}\sum_{j\in H_i}\Big[\sum_{k=1}^{i^2} x(F_{ik})\Big]\Big[\frac{1}{i^2}\sum_{k=1}^{i^2}(\xi-1)_k^{M_i}(G_j)\Big] \\
&\leq \sum_{i\in I}\sum_{j\in H_i}\Big[\frac{1}{i^2}\sum_{k=1}^{i^2} x(F_{ik})\Big]\xi, \text{ by Lemma 2.3,} \\
&\leq \xi\sum_{i\in I}\sum_{j\in H_i}\frac{1}{|H_i|}\sum_{k\in R_i} x(F_{ik}) \\
&\leq \xi\sum_{i\in I}\sum_{k\in R_i} x(F_{ik}) \\
&\leq \xi x(\cup_{i\in I,\, k\in R_i} F_{ik}) \leq 2\xi.
\end{aligned}$$

The latter inequality follows since $\|x\|_\xi \leq 1$ and $\cup_{i\in I\setminus\{\min I\},\, k\in R_i} F_{ik} \in S_\xi$. Indeed, the cardinality of the set $\{\min F_{ik} : k \in R_i,\, i \in I\}$ does not exceed that of J since $|R_i| = |H_i|$, for all $i \in I$. It follows now, since $|J| \leq \min G_1$, that $\{\min F_{ik} : k \in R_i,\, i \in I \setminus \{\min I\}\}$ belongs to S_1 and thus $\cup_{i\in I,\, k\in R_i} F_{ik}$ is the union of two members of S_ξ. Concluding,

$$\begin{aligned}
\Big(\sum_{i\in I} x(F_i)u_i\Big)(\cup_{j\in J} G_j) &= \sum_{i\in I}\sum_{j\in J_i\setminus H_i} x(F_i)u_i(G_j) + \sum_{i\in I}\sum_{j\in H_i} x(F_i)u_i(G_j) \\
&\leq 4\xi + 2\xi = 6\xi.
\end{aligned}$$

Lemma 4.1 now implies that $(\sum_{i=1}^{p} x(F_i)u_i)(G) \leq 18\xi$, for every $p \in \mathbb{N}$ and $x \in c_{00}$, $\|x\|_\xi \leq 1$, with $x(\{i\}) \geq 0$, $i \in \mathbb{N}$. It follows that $\|P\| \leq 18\xi$. The proof of the proposition is now complete. □

Proposition 4.7. *Let $1 \leq \xi < \omega$ and (u_n) be a block basis of (e_n^ξ) satisfying the following*

1. *$u_n = v_n + w_n$ with $supp v_n \cap supp w_n = \emptyset$, $n \in \mathbb{N}$.*
2. *(w_n) is equivalent to the unit vector basis of c_0.*
3. *$\lim_n \|v_n\|_\xi = 0$ yet $\sup_n \|\sum_{i=1}^{n} v_i\|_\xi = \infty$.*

Then there exists no projection from X^ξ onto the closed linear span of (u_n).

Proof. Let X denote the closed linear span of (u_n) in X^ξ and assume that $P\colon X^\xi \to X$ is a bounded linear projection. Note that since (e_n^ξ) is unconditional our assumptions yield that (u_n) is semi-normalized in X^ξ. Lemma 2.a.11 of [11] now yields that (w_n) dominates (v_n) contradicting 3. as $\sup_n \|\sum_{i=1}^{n} w_i\|_\xi < \infty$. □

It is easy to construct a normalized convex block basis of (e_n^ξ), $\xi \geq 1$, satisfying conditions 1.-3. Indeed, let $M \in [\mathbb{N}]$, $M = (m_n)$, such that $\sum_n \frac{1}{m_n} < \infty$. Let $q_n = \min F_n^\xi(M)$, $n \in \mathbb{N}$ (recall that $F_n^\xi(M) = supp\xi_n^M$). Because $\xi \geq 1$, $(e_{q_n}^\xi)$ is not dominated by the unit vector basis of c_0.

It follows that there exists a sequence of positive scalars (a_n) such that $\lim_n a_n = 0$ and $\sup_n \| \sum_{i=1}^n a_i e_{q_i}^\xi \| = \infty$. Set $v_n = a_n e_{q_n}^\xi$ and $w_n = \frac{1-a_n}{1-\xi_n^M(q_n)} \sum_{i \in F_n^\xi(M) \setminus \{q_n\}} \xi_n^M(i) e_i^\xi$, $n \in \mathbb{N}$. Finally, we let $u_n = v_n + w_n$, $n \in \mathbb{N}$. Evidently, (u_n) is a normalized convex block basis of (e_n^ξ) satisfying 1. and 3. It remains to show that 2. holds. We observe that since $\sum_n \frac{1}{m_n} < \infty$ and (ξ_n^M) is equivalent to the unit vector basis of c_0, then letting $x_n = \sum_{i \in F_n^\xi(M) \setminus \{q_n\}} \xi_n^M(i) e_i^\xi$, we have that $\sup_n \| \sum_{i=1}^n x_i \|_\xi < \infty$. It follows that (w_n) is equivalent to the unit vector basis of c_0 as $\lim_n a_n = 0$ and $\lim_n \xi_n^M(q_n) = 0$.

References

[1] D. Alspach and S.A. Argyros, *Complexity of weakly null sequences*, Dissertationes Mathematicae, **321**, (1992), 1–44.

[2] G. Androulakis and E. Odell, *Distorting mixed Tsirelson spaces*, Israel J. Math. **109** (1999), 125–149.

[3] S.A. Argyros and I. Deliyanni, *Examples of asymptotic l_1 Banach spaces*, Trans. Amer. Math. Soc. **349**, (1997), 973–995.

[4] S.A. Argyros, S. Mercourakis and A. Tsarpalias, *Convex unconditionality and summability of weakly null sequences*, Israel J. Math. **107** (1998), 157–193.

[5] S.A. Argyros and V. Felouzis, *Interpolating hereditarily indecomposable Banach spaces*, (preprint).

[6] S.A. Argyros and I. Gasparis, *Unconditional structures of weakly null sequences*, (preprint).

[7] P. Cembranos, *The hereditary Dunford-Pettis property on $C(K, E)$*, Ill. J. Math. **31** (1987), 365–373.

[8] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proc. A.M.S., (to appear).

[9] A. Kechris, *Classical Descriptive Set Theory*, GTM Vol. 156, Springer-Verlag, Berlin (1994).

[10] K. Kuratowski, *Applications of the Baire-category method to the problem of independent sets*, Fund. Math. **81** (1973), 65–72.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, II*, Springer-Verlag, Berlin.

[12] J. Mycielski, *Almost every function is independent*, Fund. Math. **81** (1973), 43–48.

[13] E. Odell, *On quotients of Banach spaces having shrinking unconditional bases*, Ill. J. Math. **36** (1992), 681–695.

[14] E. Odell, N. Tomczak-Jaegermann, R. Wagner *Proximity to ℓ_1 and Distortion in asymptotic ℓ_1 spaces*, J. Funct. Anal. **150** (1997), no. 1, 101–145.

[15] J. Schreier, *Ein Gegenbeispiel zur theorie der schwachen konvergenz*, Studia Math., **2**, (1930), 58–62.

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